

# QUASI-ISOMETRY CLASSIFICATION OF RIGHT-ANGLED ARTIN GROUPS II: SEVERAL INFINITE OUT CASES

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**ABSTRACT.** We are motivated by the question that for which class of right-angled Artin groups (RAAG's), the quasi-isometry classification coincides with commensurability classification. This is previously known for RAAG's with finite outer automorphism groups. In this paper, we identify two classes of RAAG's, where their outer automorphism groups are allowed to contain adjacent transvections and partial conjugations, hence infinite. If  $G$  belongs to one of these classes, then any other RAAG  $G'$  is quasi-isometric to  $G$  if and only if  $G'$  is commensurable to  $G$ . We also show that in this case, there exists an algorithm to determine whether two RAAG's are quasi-isometric by looking at their defining graphs. Compared to the finite out case, as well as the previous quasi-isometry rigidity results for symmetric spaces, thick Euclidean buildings and mapping class groups, the main issue we need to deal with here is the reconstruction map may not have nice properties as before, or may not even exist. We introduce a deformation argument, as well as techniques from cubulation to deal with this issue.

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## 1. INTRODUCTION

**1.1. Motivation and Background.** Given a quasi-isometry  $q : X \rightarrow Y$  between two metric spaces, one way of understanding  $q$  is to specify a collection of subspaces of  $X$  and  $Y$  such that they are stable under  $q$ , and encode the coarse intersection pattern of these subspaces of  $X$  (or  $Y$ ) in a combinatorial object  $C_X$  (or  $C_Y$ ). Then  $q$  induces a “morphism”  $q_* : C_X \rightarrow C_Y$ . Then it is natural to ask the converse: whether one can reconstruct a map between  $X$  and  $Y$  from a given morphism between  $C_X$  and  $C_Y$ . Such reconstruction problems play an important role in proving rigidity properties of  $q$ . Here are two examples.

- When  $X = Y = SL(n, \mathbb{R})/SO(n)$  for  $n \geq 3$ ,  $q$  preserves the intersection pattern of maximal flats in  $X$ , hence induces an automorphism of the spherical building at infinity. However, it follows from the fundamental theorem of projective geometry that any such automorphism actually comes from a homothety of  $X$ . Then one deduces that every such  $q$  is of bounded distance from a homothety. This is a special case of the results in [KL97, EF97].
- When  $X$  and  $Y$  are the mapping class groups of oriented closed surfaces of genus  $\geq 2$ ,  $q$  preserves the intersection pattern of Dehn twist flats [Ham05, BKMM12], hence induces an automorphism of the curve complex. However, Ivanov’s theorem tells us any such automorphism is induced by a mapping class, hence  $q$  is of bounded distance from a left multiplication.

Similar scheme appears in the study of quasi-isometries between right-angled Artin groups (RAAG). In certain cases, one can reconstruct a map between RAAG’s from a given isomorphism of the associated right-angled buildings or extension complexes [BKS08, Hua14a]. However, the situation is different from the above cases in the following aspects:

- (1) The existence of the reconstruction map relies on strong assumption of the outer automorphism groups of the RAAG’s. One can easily find an interesting case where such reconstruction map can not exist.
- (2) Even if the reconstruction map exists, it may not be as nice as before. This is due to the fact that RAAG’s may not “branch” as much as symmetric spaces, thick Euclidean buildings or mapping class groups. Extra conditions are needed to make the reconstruction map “nice”, and such cases are studied in [BKS08, Hua14a].

One goal of this paper is to deal with the above issues. We will first identify the largest class of RAAG’s such that the reconstruction map always exists, and study their rigidity properties. Then we will introduce another class of RAAG’s, where the reconstruction map fails to exist, and indicate how to get around this issue. It turns out that ideas from cubulation are relevant.

The previous quasi-isometry classification results of RAAG’s fall into two classes with strong contrast in their conclusions. [BN08] identifies a class of RAAG’s whose quasi-isometry type does not depend on the defining graph, while [BKS08] identifies another class of RAAG’s such that two RAAG’s in this class are quasi-isometric if and only if they are isomorphic. Higher dimensional generalizations of these two

cases are in [BJN10] and [Hua14a] respectively. We intent to understand this strong contrast by “interpolating” between these two cases. And the classes of RAAG’s discussed in this paper sever as an initial step towards this goal.

**1.2. Main results and open questions.** We denote the RAAG with defining graph  $\Gamma$  by  $G(\Gamma)$ . Our search for appropriate classes of RAAG’s is roughly guided by the outer automorphism group  $\text{Out}(G(\Gamma))$ . Namely if a property is true for all elements in  $\text{Out}(G(\Gamma))$ , then we ask whether it is also true for all quasi-isometries of  $G(\Gamma)$ . See Section 2.3 for a review of  $\text{Out}(G(\Gamma))$ . Since we are mainly interested in the case where  $\text{Out}(G(\Gamma))$  is infinite, we need to focus on the 3 types of generators of  $\text{Out}(G(\Gamma))$  which are of infinite order, namely the adjacent transvections, non-adjacent transvections and partial conjugations. Adjacent transvection happens inside a free Abelian subgroup, so it has relatively nice behaviour compared other types. We deal with it first.

**Definition 1.1.**  $G(\Gamma)$  is of *weak type I* if

- (1)  $\Gamma$  is connected and does not contain any separating closed star.
- (2) There does not exist vertices  $v, w \in \Gamma$  such that  $d(v, w) = 2$  and  $\Gamma = St(v) \cup St(w)$ .

We caution the reader that in this paper, the closed star of a vertex  $v$ , which we denote by  $St(v)$ , is defined to be the full subgraph spanned by  $v$  and vertices adjacent to  $v$ . This definition is slightly different from the usual one. Similarly,  $lk(v)$  is defined to be the full subgraph spanned by vertices adjacent to  $v$ .

It turns out that  $G(\Gamma)$  is of weak type I if and only if the reconstruction map for  $G(\Gamma)$  always exists, see Theorem 3.31 for a precise statement. In particular, all RAAG’s with finite outer automorphism group are of weak type I.

A simple example of RAAG of weak type I can be obtained by taking  $\Gamma$  to be the graph obtained by gluing a 5-cycle and a 3-cycle along an edge. If  $G(\Gamma)$  is of weak type I, then  $\text{Out}(G(\Gamma))$  does not contain non-adjacent transvections and partial conjugations, however, adjacent transvections are allowed.

**Theorem 1.2.** *If  $G(\Gamma)$  and  $G(\Gamma')$  are of weak type I, then they are quasi-isometric if and only if they are isomorphic.*

Having weak type I is not a quasi-isometric invariant. However, the following weaker version of Theorem 1.2 is true when only  $G(\Gamma_1)$  is of weak type II.

**Theorem 1.3.** *Suppose  $G(\Gamma_1)$  is of weak type I. Then the following are equivalent:*

- (1)  $G(\Gamma_2)$  is quasi-isometric to  $G(\Gamma_1)$ .
- (2)  $G(\Gamma_2)$  is isomorphic to a subgroup of finite index in  $G(\Gamma_1)$ .
- (3)  $G(\Gamma_2)$  is isomorphic to a special subgroup of  $G(\Gamma_1)$ .

We refer to Section 2.4 for the definition of special subgroups.

**Remark 1.4.**

- (1) If  $\text{Out}(G(\Gamma))$  is finite, then all finite index RAAG subgroups of  $G(\Gamma)$  are special subgroups ([Hua14a, Theorem 1.4]). However, in the case of RAAG’s of weak type I, though all finite index RAAG subgroups are isomorphic to a special subgroup, they may not be special subgroups themselves (considering finite index subgroups of  $\mathbb{Z} \oplus \mathbb{Z}$ ). This suggests the lost of rigidity when passing to larger outer automorphism group.

- (2) [Hua14a, Theorem 1.3] suggests that  $G(\Gamma_2)$  is quasi-isometric to  $G(\Gamma_1)$  if and only if their extension complexes (Section 2.3) are isomorphic, given  $\text{Out}(G(\Gamma_1))$  is finite. The if only direction is still true in the case of weak type I group, but the other direction is not clear.

Next we deal with partial conjugations. Since they comes from separating closed stars in  $\Gamma$ , one may want to cut  $\Gamma$  into good pieces along separating closed stars, however, this is not well-defined in general. Then one may try the opposite way and look at graphs obtained by gluing good pieces along vertex stars in a nice way. By studying such examples, we identify the following class of RAAG's.

**Definition 1.5.**  $G(\Gamma)$  is of *type II* if  $\Gamma$  is connected and for every pair of distinct vertices  $v, w \in \Gamma$ ,  $lk(v) \cap lk(w)$  does not separate  $\Gamma$ .

This condition has a geometric interpretation.  $lk(v)$  corresponds to hyperplanes in the universal covering of the Salvetti complex, so  $lk(v) \cap lk(w)$  corresponds to the intersection of hyperplanes. Definition 1.5 can be roughly interpreted as “hyperplanes of codimension 2 do not separate”.

A model example is taking  $\Gamma$  to be the union of a 5-cycle and a 6-cycle identified along a closed vertex star. If  $G(\Gamma)$  is of type II, then  $\text{Out}(G(\Gamma))$  may contain partial conjugations and adjacent transvections, but not non-adjacent transvections.

A similar but different condition, called SIL, has been studied in [CRSV10].

**Theorem 1.6.** *If  $G(\Gamma_1)$  is a right-angled Artin group of type II, then  $G(\Gamma_2)$  is quasi-isometric to  $G(\Gamma_1)$  if and only if  $G(\Gamma_2)$  is commensurable to  $G(\Gamma_1)$ . Moreover, there exists a right-angled Artin group  $G(\Gamma)$  such that  $G(\Gamma_1)$  and  $G(\Gamma_2)$  are isomorphic to special subgroups in  $G(\Gamma)$ .*

The following is a consequence of Theorem 1.3, Theorem 1.6 and [Hua14a, Section 6.3].

**Corollary 1.7.** *Let  $G(\Gamma)$  be a right-angled Artin group of type II or weak type I. Then there is an algorithm to determine whether a given right-angled Artin group  $G(\Gamma')$  is quasi-isometric to  $G(\Gamma)$  or not.*

We close this section with several comments and open questions. A RAAG of weak type I is not necessarily of type II. The following class contains RAAG's of both weak type I and type II (see Lemma 3.28).

**Definition 1.8.**  $G(\Gamma)$  is said to have weak type II if  $\Gamma$  is connected and for vertices  $v, w \in \Gamma$  such that  $d(v, w) = 2$ ,  $\Gamma \setminus (lk(v) \cap lk(w))$  is connected.

It turns out that weak type II is a quasi-isometry invariant for RAAG's, see Corollary 3.24. Though a large portion of our discussion also generalize to RAAG's of weak type II, the following question remains open.

**Question 1.9.** *Suppose  $G(\Gamma)$  is of weak type II and  $G(\Gamma')$  is quasi-isometric to  $G(\Gamma)$ . Is  $G(\Gamma')$  commensurable to  $G(\Gamma)$ ?*

The techniques in this paper does not seem to apply effectively to the case when there are non-adjacent transvections in the outer automorphism group. Indeed, in this case, there is serious breakdown of the above form of rigidity. For example, there exist two tree RAAG's which are quasi-isometric but not commensurable ([BN08]). This leads to the following question:

**Question 1.10.** *Suppose  $\Gamma$  is connected and  $\text{Out}(G(\Gamma))$  contains non-trivial non-adjacent transvection. Does there exist  $\Gamma'$  such that  $G(\Gamma)$  and  $G(\Gamma')$  are quasi-isometric, but not commensurable?*

**1.3. Comments on the Proof.** We refer to Section 2.3 for definitions of relevant terms. The Salvetti complex of  $G(\Gamma)$  is denoted by  $S(\Gamma)$ , the universal covering of  $S(\Gamma)$  is denoted by  $X(\Gamma)$ , and flats in  $X(\Gamma)$  that cover standard tori in  $S(\Gamma)$  are called standard flats. Two standard flats are *coarsely equivalent* if they have finite Hausdorff distance. Let  $\mathcal{P}(\Gamma)$  be the extension complex of  $X(\Gamma)$ . The  $k$ -dimensional simplices in  $\mathcal{P}(\Gamma)$  are in 1-1 correspondence with coarse equivalent classes of  $(k+1)$ -dimensional standard flats in  $X(\Gamma)$ . Thus  $\mathcal{P}(\Gamma)$  captures the coarse intersection pattern of standard flats in  $X(\Gamma)$ . It turns out to be a quasi-isometry invariant for a large class of RAAG's.

**Theorem 1.11.** *Let  $q : G(\Gamma_1) \rightarrow G(\Gamma_2)$  be a quasi-isometry. Suppose  $\text{Out}(G(\Gamma_i))$  does not contain any non-adjacent transvection for  $i = 1, 2$ . Then  $q$  preserves maximal standard flats up to finite Hausdorff distance. Hence induces a simplicial isomorphism  $q_* : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$ .*

The assumption of Theorem 1.11 are motivated by the fact that any automorphism of  $G(\Gamma)$  preserves maximal standard flats up to finite Hausdorff distance if and only if there is no non-adjacent transvection in  $\text{Out}(G(\Gamma))$ .

One can try to reconstruct a “straightening” of  $q$  from  $q_*$  as follows. Pick vertex  $x \in X(\Gamma_1)$  and let  $\{F_i\}_{i \in I}$  be the collection of maximal standard flats containing  $x$ . Under mild condition we have  $x = \cap_{i \in I} F_i$ . Each  $F_i$  is associated with a maximal standard flat  $F'_i \subset X(\Gamma_2)$  by Theorem 1.11. It is natural to define  $\bar{q} : G(\Gamma_1) \rightarrow G(\Gamma_2)$  such that  $\bar{q}(x) = \cap_{i \in I} F'_i$ . However, it is possible that  $\cap_{i \in I} F'_i = \emptyset$ .

**1.3.1. The weak type I case.** It turns out that this is exactly the case that we always have  $\cap_{i \in I} F'_i \neq \emptyset$ . Under mild condition  $\cap_{i \in I} F'_i \neq \emptyset$  is actually a point. Then the map  $\bar{q}$  is well-defined, and it preserves all the maximal standard flats. A priori,  $\bar{q}$  may not preserve standard flats which are not maximal, and the key to prove Theorem 1.3 is to deform  $\bar{q}$  such that it preserves all standard flats.

A standard flat is *rigid* if  $\bar{q}$  will send its vertex set to the vertex set of another standard flat, otherwise it is *non-rigid*. For example, all intersections of maximal standard flats are rigid, but the converse may not be true.

We will deform  $\bar{q}$  in an inductive way. The first step is to show one can deform with respect to minimal rigid flats such that any standard flat contained in a minimal rigid flat is preserved by  $\bar{q}$ . To continue the induction argument, note that inside a (not necessarily minimal) rigid flat, there are directions which are rigid and directions which are not rigid. So we need to perform the deformation such that each move does not mess up the previous moves, and does not place obstructions to the moves after. The second point is non-trivial, since rigid flats may intersect each other in a complicated pattern. To describe the deformation, we introduce an atlas for  $G(\Gamma)$ , where the vertex sets of standard flats are consistently labelled by free Abelian groups. The detail is discussed in Section 4.

**1.3.2. The type II case.** In this case, the map  $\bar{q}$  may fail to exist. For example, one can take  $q$  to be a partial conjugation.

Instead of reconstruct maps, we ask whether one can reconstruct the space  $X(\Gamma)$  from  $\mathcal{P}(\Gamma)$ . Note that  $X(\Gamma)$  is a  $CAT(0)$  cube complex. In general, the collection

of halfspaces in a  $CAT(0)$  cube complex, and their intersection pattern contain the complete information needed to reconstruct the complex itself. This can be formalized in the language of pocset (see Definition 2.7 and Theorem 2.9).

Then we ask whether we can put a pocset structure on  $\mathcal{P}(\Gamma)$  such that it is the right one to recover  $X(\Gamma)$ . This can be always done. Roughly speaking, one can embed  $\mathcal{P}(\Gamma)$  into the Tits boundary of  $X(\Gamma)$ . And the collection of subsets of  $\mathcal{P}(\Gamma)$  which are the intersections of  $\mathcal{P}(\Gamma)$  and the Tits boundary of halfspaces of  $X(\Gamma)$  has a natural pocset structure.

Briefly speaking,  $X(\Gamma)$  is equivalent to  $\mathcal{P}(\Gamma)$  with some decorations on  $\mathcal{P}(\Gamma)$ . In general, these decoration depend on how one embeds  $\mathcal{P}(\Gamma)$  into the Tits boundary, so they do not come from intrinsic properties of  $\mathcal{P}(\Gamma)$ . Thus the rigidity of  $X(\Gamma)$  depends on the amount of non-intrinsic decorations we need to put on  $\mathcal{P}(\Gamma)$ . For example, in the most rigid case when  $G(\Gamma)$  has finite outer automorphism group, the amount of decorations needed is minimal. The worst case is when  $X(\Gamma)$  is tree, then  $\mathcal{P}(\Gamma)$  is just a discrete set.

If  $G(\Gamma)$  is of type II, then the amount of extra decoration is reasonably small (see Corollary 3.14 (1) for a precise statement). We prove Theorem 1.6 in two steps. The pocset structure on  $\mathcal{P}(\Gamma)$  is defined in terms of certain partition of  $\mathcal{P}(\Gamma)$ . First we show it is possible to refine this partition to obtain a new pocset which does not admit any reasonable further refinement. It turns out the new pocset gives rise to another RAAG which is commensurable to the original one. Such RAAG's are called *prime* RAAG's (Definition 5.3). Then we show two prime RAAG's are quasi-isometric if and only if they are isomorphic, which finishes the proof of Theorem 1.6. We caution the reader that in order to avoid some technicality, we work with pocset on  $\Gamma$  rather than  $\mathcal{P}(\Gamma)$  in Section 5. However, the idea is similar.

**1.4. Organization of the Paper.** In Section 2 we summary and generalize several results from [Hua14a] about  $CAT(0)$  cube complex, right-angled Artin group and extension complex. In particular, Theorem 1.11 will be proved in Section 2.2.

In Section 3 we study the structure of the extension complex  $\mathcal{P}(\Gamma)$  and prove Theorem 1.2 at the end of Section 3.

The goal of Section 4 is to prove Theorem 1.3. We will introduce a notion of atlas for right-angled Artin group in Section 4.1 and use this in Section 4.2 as an effective language to describe the deformation argument mentioned above.

We prove Theorem 1.6 in Section 5. Section 5 does not depend on Section 4.

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## 2. PRELIMINARIES

**2.1. Notations and conventions.** Notations here are consistent with [Hua14a, Section 2.1]. All graphs in this paper are simplicial. The flag complex of a graph  $\Gamma$  is denoted by  $F(\Gamma)$ , i.e.  $F(\Gamma)$  is a flag complex such that its 1-skeleton is  $\Gamma$ .

Let  $K$  be a polyhedral complex.

- (1) By viewing the 1-skeleton of  $K$  as a metric graph with edge lengths 1, we obtain a metric defined on the 0-skeleton of  $K$ , which we denote by  $d$ .
- (2) A subcomplex  $K' \subset K$  is *full* if  $K'$  contains all the subcomplexes of  $K$  which have the same vertex set as  $K'$ . If  $K$  is 1-dimensional, then we also call  $K'$  a *full subgraph*.

- (3) We use  $\circ$  to denote the join of two graphs, and  $*$  to denote the join of two polyhedral complexes.
- (4) For a set of vertices  $V \subset K$ ,  $V^\perp$  is defined to be collection of vertices which are adjacent to each vertex in  $V$ .
- (5) Let  $v \in K$  be a vertex. The *link* of  $v$  in  $K$ , denoted by  $lk(v, K)$  or  $lk(v)$  when  $K$  is clear, is defined to be the full subcomplex spanned by  $v^\perp$ . The *closed star* of  $v$  in  $K$ , denoted by  $St(v, K)$  or  $St(v)$  when  $K$  is clear, is defined to be the full subcomplex spanned by  $v$  and  $v^\perp$ .
- (6) Let  $M \subset K$  be an arbitrary subset. We denote the collection of vertices inside  $M$  by  $v(M)$ .

We will use the following simple observation repeatedly.

**Lemma 2.1.** *Let  $K$  be a simplicial complex and let  $K^{(1)}$  be the 1-skeleton of  $K$ . Suppose  $L \subset K$  be a full subcomplex. Then there is a 1-1 correspondence between connected components of  $K \setminus L$  and  $K^{(1)} \setminus L^{(1)}$ . Moreover, the intersection of each component of  $K \setminus L$  with  $K^{(1)}$  is a component of  $K^{(1)} \setminus L^{(1)}$ .*

Let  $X$  be a metric space. We use  $d_H$  to denote the Hausdorff distance and use  $N_R(Y)$  to denote the  $R$ -neighbourhood of a subspace  $Y \subseteq X$ . Two subspaces are *coarsely equivalent* if they have finite Hausdorff distance. A subspace  $V \subseteq X$  is the *coarse intersection* of subspaces  $Y_1$  and  $Y_2$  if  $V$  is at finite Hausdorff distance from  $N_R(Y_1) \cap N_R(Y_2)$  for all sufficiently large  $R$ . In general the coarse intersection of two subspaces might not exist.

**2.2.  $CAT(0)$  space and  $CAT(0)$  cube complex.** We mention several relevant facts here and refer to [BH99] and [Sag12] for more background on  $CAT(0)$  spaces and  $CAT(0)$  cube complexes. The reader can also check [Hua14a, Section 2.2].

Let  $(X, d)$  be a  $CAT(0)$  space and let  $C \subset X$  be a convex subset. We denote the nearest point projection from  $X$  to  $C$  by  $\pi_C : X \rightarrow C$ . Denote the Tits boundary of  $X$  by  $\partial_T X$ . If  $C' \subset X$  be another convex set, then  $C'$  is *parallel* to  $C$  if  $d(\cdot, C)|_{C'}$  and  $d(\cdot, C')|_C$  are constant functions. We define the *parallel set* of  $C$ , denoted by  $P_C$ , to be the union of all convex subsets of  $X$  parallel to  $C$ .

Now we turn to  $CAT(0)$  cube complexes. All cube complexes in this paper are assumed to be finite dimensional. There are two common metrics on a  $CAT(0)$  cube complex, namely the  $CAT(0)$  metric and the  $l^1$ -metric. In this paper, we will mainly use the  $CAT(0)$  metric unless otherwise specified.

A *geodesic segment*, *geodesic ray* or *geodesic* in a  $CAT(0)$  cube complex  $X$  is an isometric embedding of  $[a, b]$ ,  $[0, \infty)$  or  $\mathbb{R}$  into  $X$  with respect to the  $CAT(0)$  metric. A *combinatorial geodesic segment*, *combinatorial geodesic ray* or *combinatorial geodesic* is a  $l^1$ -isometric embedding of  $[a, b]$ ,  $[0, \infty)$  or  $\mathbb{R}$  into  $X^{(1)}$  such that its image is a subcomplex.

The collection of convex subcomplexes in a  $CAT(0)$  cube complex enjoys the following version of Helly's property ([Ger98]):

**Lemma 2.2.** *Let  $X$  be as above and  $\{C_i\}_{i=1}^k$  be a collection of convex subcomplexes. If  $C_i \cap C_j \neq \emptyset$  for any  $1 \leq i \neq j \leq k$ , then  $\cap_{i=1}^k C_i \neq \emptyset$ .*

**Lemma 2.3.** [Hag08] *Let  $X$  be a  $CAT(0)$  cube complex and let  $Y \subset X$  be a convex subcomplex. Then  $Y$  is also combinatorially convex in the sense that any combinatorial geodesic segment joining two vertices in  $Y$  is contained in  $Y$ .*

**Lemma 2.4** (Lemma 2.10 of [Hua14b]). *Let  $X$  be a  $CAT(0)$  cube complex of dimension  $n$  and let  $C_1, C_2$  be convex subcomplexes. Denote  $\Delta = d(C_1, C_2)$ . Put  $Y_1 = \{y \in C_1 \mid d(y, C_2) = \Delta\}$  and  $Y_2 = \{y \in C_2 \mid d(y, C_1) = \Delta\}$ . Then:*

- (1)  $Y_1$  and  $Y_2$  are not empty.
- (2)  $Y_1$  and  $Y_2$  are convex;  $\pi_{C_1}$  map  $Y_2$  isometrically onto  $Y_1$  and  $\pi_{C_2}$  map  $Y_1$  isometrically onto  $Y_2$ ; the convex hull of  $Y_1 \cup Y_2$  is isometric to  $Y_1 \times [0, \Delta]$ .
- (3)  $Y_1$  and  $Y_2$  are subcomplexes, and  $\pi_{C_2}|_{Y_1}$  is a cubical isomorphism with its inverse given by  $\pi_{C_1}|_{Y_2}$ .
- (4) there exist  $A = A(\Delta, n, \epsilon)$  such that if  $p_1 \in C_1$ ,  $p_2 \in C_2$  and  $d(p_1, Y_1) \geq \epsilon > 0$ ,  $d(p_2, Y_2) \geq \epsilon > 0$ , then

$$(2.5) \quad d(p_1, C_2) \geq \Delta + Ad(p_1, Y_1), d(p_2, C_1) \geq \Delta + Ad(p_2, Y_2)$$

The above lemma implies  $Y_1$  and  $Y_2$  are coarsely equivalent, and  $Y_1$  (or  $Y_2$ ) is the coarse intersection of  $C_1$  and  $C_2$ . We use  $\mathcal{I}(C_1, C_2) = (Y_1, Y_2)$  to describe this situation, where  $\mathcal{I}$  stands for “intersect”.

The cubical structure of  $X$  naturally gives rise to a collection of cubical tracks in  $X$ , which are called *hyperplanes*. Each hyperplane separates  $X$  into exactly two *halfspaces*. Pick edge  $e \subset X$ , the hyperplane dual to  $e$  is defined by  $\pi_e^{-1}(m)$  where  $m$  is the middle point of  $e$ . The sets  $Y_1$  and  $Y_2$  in Lemma 2.4 can be characterized in terms of hyperplanes.

**Lemma 2.6.** ([Hua14a, Lemma 2.6]) *Let  $X, C_1, C_2, Y_1$  and  $Y_2$  be as in Lemma 2.4. Pick an edge  $e$  in  $Y_1$  (or  $Y_2$ ) and let  $h$  be the hyperplane dual to  $e$ . Then  $h \cap C_i \neq \emptyset$  for  $i = 1, 2$ . Conversely, if a hyperplane  $h'$  satisfies  $h' \cap C_i \neq \emptyset$  for  $i = 1, 2$ , then  $\mathcal{I}(h' \cap C_1, h' \cap C_2) = (h' \cap Y_1, h' \cap Y_2)$  and  $h'$  comes from the dual hyperplane of some edge  $e'$  in  $Y_1$  (or  $Y_2$ ).*

The collection of halfspaces in  $X$  contains enough information to recover  $X$ . More generally, we can view  $X$  as a space with walls, and [Sag95, HP98] introduces a way to construct a  $CAT(0)$  cube complex from a given space with walls. There are several variations and developments of this construction ([Rol98, Nic04, CN05, HW14]). Here we follow the construction in [Rol98], see Sageev’s notes [Sag12].

**Definition 2.7** (Definition 1.5 of [Sag12]). A *pocset* is a partially ordered set with an involution  $A \rightarrow A^c$  such that

- (1)  $A \neq A^c$  and  $A$  and  $A^c$  are incomparable.
- (2)  $A \leq B$  implies  $B^c \leq A^c$ .

Note that the collection of all closed halfspaces in a  $CAT(0)$  cube complex forms a pocset (the partially order comes from inclusion of sets).

**Definition 2.8** (Definition 2.1 of [Sag12]). Let  $P$  be a pocset. An *ultrafilter*  $U$  is a subset of  $P$  such that

- (1) For all pairs  $\{A, A^c\}$  in  $P$ , precisely one of them is in  $U$ .
- (2) If  $A \in U$  and  $A \leq B$ , then  $B \in U$ .

For example, pick a vertex  $p$  in a  $CAT(0)$  cube complex  $X$ , then the collection of closed halfspaces in  $X$  that contains  $p$  forms an ultrafilter. Note that if  $U$  is an ultrafilter and  $A$  is minimal in  $U$  with respect to the partial order on  $P$ , then  $(U \setminus \{A\}) \cup \{A^c\}$  is also an ultrafilter.



**Theorem 2.9** ([Rol98]). *If  $P$  is a finite pocset, then there is a  $CAT(0)$  cube complex  $X$  such that its vertices are in 1-1 correspondence with ultrafilters of  $P$  and two vertices  $U$  and  $U'$  are connected by an edge if and only if  $U' = (U \setminus \{A\}) \cup \{A^c\}$  for some  $A$  minimal in  $U$ . Moreover, there is a natural pocset isomorphism from  $P$  to the pocset of halfspaces in  $X$ .*

If  $P$  is infinite, then similar conclusions hold under the additional assumptions that  $P$  is discrete and of finite width, see [Rol98, Sag12].

**2.3. Basics about RAAG's.** Pick a finite simplicial graph  $\Gamma$ , let  $G(\Gamma)$  be the right-angled Artin group (RAAG) with defining graph  $\Gamma$  and let  $S(\Gamma)$  be the Salvetti complex of  $G(\Gamma)$ . Denote the universal cover of  $S(\Gamma)$  by  $X(\Gamma)$ . Pick a standard generating set  $S$  for  $G(\Gamma)$ , we label the 1-cells in  $S(\Gamma)$  by elements in  $S$  and choose an orientation for each 1-cell in  $S(\Gamma)$ . This lifts to orientation and labelling of edges in  $X(\Gamma)$  which are invariant under the action  $G(\Gamma) \curvearrowright X(\Gamma)$ .

Let  $\Gamma' \subset \Gamma$  be a full subgraph. Then the images of the embeddings  $G(\Gamma') \rightarrow G(\Gamma)$  and  $S(\Gamma') \rightarrow S(\Gamma)$  are called *standard subgroup* and *standard subcomplex* respectively. *Standard subcomplexes* of  $X(\Gamma)$  are lifts of standard subcomplexes of  $S(\Gamma)$ . When  $\Gamma'$  is a complete subgraph,  $G(\Gamma')$  is called a *standard Abelian subgroup*,  $S(\Gamma')$  is called a *standard torus*, and lifts of  $S(\Gamma')$  are called *standard flats*. One dimensional standard flats are also called *standard geodesics*.

**Definition 2.10.** For every edge  $e \in X(\Gamma)$ , there is a vertex in  $\Gamma$  which shares the same label as  $e$ . We denote this vertex by  $V_e$ . If  $K \subset X(\Gamma)$  is a subcomplex ( $K$  does not need to be a standard subcomplex), we define  $V_K$  to be  $\{V_e \mid e \text{ is an edge in } K\}$  and  $\Gamma_K$  to be the full subgraph spanned by  $V_K$ .  $\Gamma_K$  is called the *support* of  $K$ . Pick a vertex  $v \in X(\Gamma)$  and a full subgraph  $\Gamma' \subset \Gamma$ , we denote the unique standard subcomplex with defining graph  $\Gamma'$  that contains  $v$  by  $K(v, \Gamma')$ .

The following two results are strengthened version of Lemma 2.4 and Lemma 2.6 in the case of coarse intersection of two standard subcomplexes:

**Lemma 2.11.** ([Hua14a, Lemma 3.1]) *Let  $\Gamma$  be a finite simplicial graph and let  $K_1, K_2$  be two standard subcomplexes of  $X(\Gamma)$ . If  $(Y_1, Y_2) = \mathcal{I}(K_1, K_2)$ , then  $Y_1$  and  $Y_2$  are also standard subcomplexes.*

We can compute the supports of  $Y_1$  and  $Y_2$  as follows.

**Lemma 2.12.** ([Hua14a, Corollary 3.2]) *Let  $K_1, K_2, Y_1$  and  $Y_2$  be as above. Let  $h$  be a hyperplane separating  $K_1$  and  $K_2$  and let  $e$  be an edge dual to  $h$ . Then  $V_e \in V_{Y_1}^\perp = V_{Y_2}^\perp$  (see Definition 2.10 for relevant notations). In particular, a vertex  $v \in \Gamma$  satisfies  $v \in V_{Y_1}$  if and only if*

- (1)  $v \in V_{K_1} \cap V_{K_2}$ .
- (2) For any hyperplane  $h'$  separating  $K_1$  from  $K_2$  and any edge  $e'$  dual to  $h'$ ,  $d(v, V_{e'}) = 1$ .

Let  $\mathcal{P}(\Gamma)$  be the extension complex of  $\Gamma$  (see [Hua14a, Section 4.1]), which is the flag complex of the extension graph introduced in [KK13]. Here is an alternative definition. The vertices of  $\mathcal{P}(\Gamma)$  are in 1-1 correspondence with the parallel classes of standard geodesics in  $X(\Gamma)$  (two standard flats are in the same parallel class if they are parallel). Two distinct vertices  $v_1, v_2 \in \mathcal{P}(\Gamma)$  are connected by an edge if for  $i = 1, 2$ , there is a standard geodesic  $l_i$  in the parallel class associated with  $v_i$  such that  $l_1$  and  $l_2$  span a standard 2-flat.

Note that edges in the same standard geodesics of  $X(\Gamma)$  has the same label, and edges in parallel standard geodesics also has the same label. This induces a well-defined labelling of vertices in  $\mathcal{P}(\Gamma)$  by vertices of  $\Gamma$ . And there is a label-preserving simplicial map  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$ , where  $F(\Gamma)$  is the flag complex of  $\Gamma$ . Moreover, since  $G(\Gamma) \curvearrowright X(\Gamma)$  by label-preserving cubical isomorphisms, we obtain an induced action  $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$  by label-preserving simplicial isomorphisms.

Note that each complete subgraph in the 1-skeleton of  $\mathcal{P}(\Gamma)$  gives rise to a collection of mutually orthogonal standard geodesics lines. Thus there is a 1-1 correspondence between the  $(k-1)$ -simplexes in  $\mathcal{P}(\Gamma)$  and parallel classes of standard  $k$ -flats in  $X(\Gamma)$  ([Hua14a, Section 4.1]). For standard flat  $F \subset X(\Gamma)$ , we denote the simplex in  $\mathcal{P}(\Gamma)$  associated with the parallel class containing  $F$  by  $\Delta(F)$ . For a standard subcomplex  $K \subset X(\Gamma)$ , define  $\Delta(K) := \cup_{\lambda \in \Lambda} \Delta(F_\lambda)$ , here  $\{F_\lambda\}_{\lambda \in \Lambda}$  is the collection of standard flats in  $K$ .

Pick arbitrary vertex  $p \in X(\Gamma)$ , one can obtain a simplicial embedding  $i_p : F(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  by considering the collection of standard flats passing through  $p$  ( $F(\Gamma)$  is the flag complex of  $\Gamma$ ). We will denote the image of  $i_p$  by  $(F(\Gamma))_p$ . Note that  $\pi \circ i_p$  is the identity map, which implies the following lemma.

**Lemma 2.13.** *The map  $i_p$  is an isometric embedding with respect to the combinatorial distance between vertices of  $F(\Gamma)$ .*

Now we look at the outer automorphism group  $\text{Out}(G(\Gamma))$  of  $G(\Gamma)$ . By [Ser89, Lau95],  $\text{Out}(G(\Gamma))$  is generated by the following four types of elements (we identify the vertex set of  $\Gamma$  with a standard generating set of  $G(\Gamma)$ ):

- (1) Given vertex  $v \in \Gamma$ , sending  $v \rightarrow v^{-1}$  and fixing all other vertices.
- (2) Graph automorphisms of  $\Gamma$ .
- (3) If  $lk(w) \subset St(v)$  for vertices  $w, v \in \Gamma$ , sending  $w \rightarrow vw$  and fixing all other vertices induces to a group automorphism. It is called a *transvection*. When  $d(v, w) = 1$ , it is an *adjacent transvection*, otherwise it is a *non-adjacent transvection*.
- (4) Suppose  $\Gamma \setminus St(v)$  is disconnected. Then one obtains a group automorphism by picking a connected component  $C$  and sending  $w \rightarrow v w v^{-1}$  for vertex  $w \in C$ . It is called a *partial conjugation*.

**2.4. Special subgroups of RAAG's.** We recall the notion of special subgroup introduced in [Hua14a]. Let  $K \subset X(\Gamma)$  be a compact convex subcomplex. Let  $\{\ell_i\}_{i=1}^s$  be a maximal collection of standard geodesics such that  $\ell_i \cap K \neq \emptyset$  for all  $i$  and  $\Delta(\ell_i) \neq \Delta(\ell_j)$  for any  $i \neq j$ . We consider the left action  $G(\Gamma) \curvearrowright X(\Gamma)$ . For each  $i$ , let  $g_i \in G(\Gamma)$  be the unique element that translates  $\ell_i$  towards the positive direction with translation length = 1 (recall that we have oriented edges of  $X(\Gamma)$  in a  $G(\Gamma)$ -invariant way). Let  $n_i = |v(K \cap c_i)|$ .

**Theorem 2.14.** ([Hua14a, Section 6.1]) *Let  $G \leq G(\Gamma)$  be the subgroup generated by  $\{g_i^{n_i}\}_{i=1}^s$ . Then*

- (1)  *$K$  is a “fundamental domain” for  $G$  in the sense that for  $g_1, g_2 \in G$ ,  $g_1 K \cap g_2 K \neq \emptyset$  if and only if  $g_1 = g_2$ . Moreover, the  $G$ -orbit of  $K$  cover the 0-skeleton of  $X(\Gamma)$ . Thus  $|G(\Gamma) : G|$  = the number of vertices in  $K$ .*
- (2) *Let  $\Gamma'$  be the 1-skeleton of the full subcomplex of  $\mathcal{P}(\Gamma)$  spanned by  $\{\Delta(\ell_i)\}_{i=1}^s$ . Then  $G$  is isomorphic to the RAAG  $G(\Gamma')$ .*

Such  $G$  is called a *special subgroup* of  $G(\Gamma)$  (associated with  $K$ ). Note that the definition of special subgroups implicitly depend on the choice of standard generators of  $G(\Gamma)$  (we can think  $G(\Gamma)$  as a fixed set, and different choices of standard generators give different ways of building  $X(\Gamma)$  from  $G(\Gamma)$ ). A subgroup is *S-special* if it is special with respect to the standard generating set  $S$ . In most part of the paper, we fix a standard generating set, so there will be no confusion.

Alternatively,  $G$  can be characterized by the fundamental group of the canonical completion ([HW08]) of the local isometry  $K \hookrightarrow X(\Gamma) \rightarrow S(\Gamma)$ . However, we will not need this fact.

Let  $G(\Gamma) \cong F_2$ , the free group with two generators. We take  $K$  to be an edge in  $X(\Gamma)$ . Then the associated special subgroup  $G$  is isomorphic to  $F_3$ . In this case, if we collapse all the  $G$ -translations of  $K$  in  $X(\Gamma)$ , then the resulting space is isomorphic to the Cayley graph for  $F_3$ . This is actually true in general.

Recall that a cellular map between cube complexes is *cubical* (see [CS11]) if its restriction  $\sigma \rightarrow \tau$  between cubes factors as  $\sigma \rightarrow \eta \rightarrow \tau$ , where the first map  $\sigma \rightarrow \eta$  is a natural projection onto a face of  $\sigma$  and the second map  $\eta \rightarrow \tau$  is an isometry.

**Theorem 2.15.** ([Hua14a, Lemma 6.13]) *There is a surjective cubical map  $q : X(\Gamma) \rightarrow X(\Gamma')$  such that*

- (1)  *$q$  maps standard flats to standard flats. Moreover, each standard flat in  $X(\Gamma')$  is the  $q$ -image of some standard flat in  $X(\Gamma)$ .*
- (2) *Pick a vertex  $x' \in X(\Gamma')$ , then  $q^{-1}(x') = g \cdot K$  for some element  $g \in G$ .*
- (3)  *$q$  is  $G$ -equivariant. In particular,  $q$  is a quasi-isometry.*

It follows from (1) that  $q$  induces a simplicial isomorphism  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$ .

**2.5. Coarse invariants for RAAG's.** Here we summarize and generalize some results from [Hua14a].

Note that every join decomposition  $\Gamma = \Gamma_1 \circ \Gamma_2$  induces a direct sum decomposition  $G(\Gamma) = G(\Gamma_1) \oplus G(\Gamma_2)$ .  $G(\Gamma)$  or  $\Gamma$  is called *irreducible* if  $\Gamma$  does not allow non-trivial join decomposition. There is a well-defined De Rham decomposition of  $X(\Gamma)$  induced by the join decomposition of  $\Gamma$ , which is stable under quasi-isometries.

**Theorem 2.16.** ([Hua14a, Theorem 2.9]) *Given  $X = X(\Gamma)$  and  $X' = X(\Gamma')$ , let  $X = \mathbb{R}^n \times \prod_{i=1}^k X(\Gamma_i)$  and  $X' = \mathbb{R}^{n'} \times \prod_{j=1}^{k'} X(\Gamma'_j)$  be the corresponding De Rham decomposition. If  $\phi : X \rightarrow X'$  is an  $(L, A)$  quasi-isometry, then  $n = n'$ ,  $k = k'$  and there exist constants  $L' = L'(L, A)$ ,  $A' = A'(L, A)$ ,  $D = D(L, A)$  such that after re-indexing the factors in  $X'$ , we have  $(L', A')$  quasi-isometry  $\phi_i : X(\Gamma_i) \rightarrow X(\Gamma'_i)$  so that  $d(p' \circ \phi, \prod_{i=1}^k \phi_i \circ p) < D$ , where  $p : X \rightarrow \prod_{i=1}^k X(\Gamma_i)$  and  $p' : X' \rightarrow \prod_{i=1}^k X(\Gamma'_i)$  are the projections.*

We are particularly interested in those standard subcomplexes that are stable under quasi-isometries.

**Definition 2.17.** A subgraph  $\Gamma_1 \subset \Gamma$  is *stable* in  $\Gamma$  if  $\Gamma_1$  is a full subgraph and for any standard subcomplex  $K \subset X(\Gamma)$  with  $\Gamma_K = \Gamma_1$  and  $(L, A)$ -quasi-isometry  $q : X(\Gamma) \rightarrow X(\Gamma')$ , there exists  $D = D(L, A, \Gamma_1, \Gamma) > 0$  and standard subcomplex  $K' \subset X(\Gamma')$  such that the Hausdorff distance  $d_H(q(K), K') < D$ . A standard subcomplex  $K \subset X(\Gamma)$  is *stable* if it arises from a stable subgraph of  $\Gamma$ .

It is clear that the intersection of two stable subgraphs is still a stable subgraph. See [Hua14a, Section 3.2] for more properties about stable subgraphs. In this paper, we will use the following two properties repeatedly:

**Lemma 2.18.** [Hua14a, Lemma 3.24] *Let  $\Gamma$  be a finite simplicial graph and pick stable subgraphs  $\Gamma_1, \Gamma_2$  of  $\Gamma$ . Let  $\bar{\Gamma}$  be the full subgraph spanned by  $V$  and  $V^\perp$  where  $V$  is the vertex set of  $\Gamma_1$ . If  $\Gamma_2 \subset \bar{\Gamma}$ , then the full subgraph spanned by  $\Gamma_1 \cup \Gamma_2$  is stable in  $\Gamma$ .*

**Lemma 2.19.** *Suppose there is no non-adjacent transvection in  $\text{Out}(G(\Gamma))$ . Then every maximal clique subgraph of  $\Gamma$  is stable.*

*Proof.* Let  $\Gamma_1 \subset \Gamma$  be a maximal clique. By [Hua14a, Theorem 3.35], it suffices to prove for any vertices  $v \in \Gamma_1$  and  $w \in \Gamma$ ,  $v^\perp \in \text{St}(w)$  implies  $w \in \Gamma_1$ . Note that  $v^\perp \in \text{St}(w)$  implies  $w \in v^\perp$  since there is no non-adjacent transvection. Then  $w$  and vertices of  $\Gamma_1$  span a clique in  $\Gamma$ , thus  $w \in \Gamma_1$  by the maximality of  $\Gamma_1$ .  $\square$

The following result is the starting point of this paper.

**Theorem 2.20.** *Let  $q : X(\Gamma_1) \rightarrow X(\Gamma_2)$  be a quasi-isometry. Suppose  $\text{Out}(G(\Gamma_i))$  does not contain any non-adjacent transvection for  $i = 1, 2$ . Then there exists a simplicial isomorphism  $q_* : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$ .*

*Proof.* Let  $\mathcal{S}(X(\Gamma_i))$  be the collection of coarse equivalence classes of stable subcomplexes in  $X(\Gamma_i)$  and let  $\mathcal{KS}(X(\Gamma_i))$  be the lattice generated by finite intersection and union of these coarse equivalence classes (the coarse intersection is well-defined by Lemma 2.11). Recall that we have a map  $\Delta$  sending standard subcomplexes of  $X(\Gamma_i)$  to subcomplexes of  $\mathcal{P}(\Gamma_i)$ , let  $\mathcal{S}(\mathcal{P}(\Gamma_i))$  be the image of  $\mathcal{S}(X(\Gamma_i))$  under  $\Delta$  and let  $\mathcal{KS}(\mathcal{P}(\Gamma_i))$  be the lattice generated by elements in  $\mathcal{S}(\mathcal{P}(\Gamma_i))$ . Then there is a lattice isomorphism:

$$\Delta : \mathcal{KS}(X(\Gamma_i)) \rightarrow \mathcal{KS}(\mathcal{P}(\Gamma_i)).$$

for  $i = 1, 2$ . Moreover, by Lemma 2.4, the quasi-isometry  $q$  induces a lattice isomorphism:

$$q_\# : \mathcal{KS}(X(\Gamma_1)) \rightarrow \mathcal{KS}(X(\Gamma_2)),$$

thus we have the following isomorphism:

$$\tilde{q}_* = \Delta \circ q_\# \circ \Delta^{-1} : \mathcal{KS}(\mathcal{P}(\Gamma_1)) \rightarrow \mathcal{KS}(\mathcal{P}(\Gamma_2)).$$

For subset  $A$  in a metric space, denote the collection of subsets which are coarsely equivalent to  $A$  by  $[A]$ . By definition, for standard flat  $F_1 \in X(\Gamma_1)$  with  $[F_1] \in \mathcal{KS}(X(\Gamma_1))$ , there exists standard flat  $F_2 \in X(\Gamma_2)$  such that  $q_\#[F_1] = [F_2]$  and  $\dim(F_1) = \dim(F_2)$ . Let  $\Phi_i^k$  be the collection of  $k$ -dimensional simplexes in  $\mathcal{KS}(\mathcal{P}(\Gamma_i))$  for  $i = 1, 2$ . Then  $\tilde{q}_*$  induces a 1-1 correspondence between  $\Phi_1^k$  and  $\Phi_2^k$  for any  $k \geq 0$ . For  $i = 1, 2$ , let  $\mathcal{V}_i^k$  be the set of vertices of simplexes in  $\cup_{m=0}^k \Phi_i^m$ . By Lemma 2.19,  $\mathcal{V}_i^{n-1}$  is exactly the 0-skeleton of  $\mathcal{P}(\Gamma_i)$ , where  $n = \dim(X(\Gamma_1)) = \dim(X(\Gamma_2))$ .

We construct a map  $q_*$  from the 0-skeleton of  $\mathcal{P}(\Gamma_1)$  to the 0-skeleton of  $\mathcal{P}(\Gamma_2)$  inductively as follows: define  $q_*(v) = \tilde{q}_*(v)$  for  $v \in \mathcal{V}_1^0$  and suppose we've already defined  $q_*$  on  $\mathcal{V}_1^k$  such that

(\*) For any simplex  $\Delta \in \mathcal{KS}(X(\Gamma_1))$ ,  $q_*$  is a bijection from  $\mathcal{V}_1^k \cap \Delta$  to  $\mathcal{V}_2^k \cap \tilde{q}_*(\Delta)$ .

Note that  $q_*$  restricted on  $\mathcal{V}_1^0$  automatically satisfies  $(*)$  for  $k = 0$ . Pick simplex  $\Delta^{k+1} \in \Phi_1^{k+1}$ , if all vertices of  $\Delta^{k+1}$  belongs to  $\mathcal{V}_1^k$ , then we do not need to do anything, otherwise pick vertex  $v \in \Delta^{k+1} \setminus \mathcal{V}_1^k$ , note that  $\Delta^{k+1}$  is the only simplex in  $\Phi_1^{k+1}$  that contains  $v$  (if there is a distinct simplex  $\Delta_1^{k+1} \in \Phi_1^{k+1}$  with  $v \in \Delta_1$ , then  $v \in \Delta_1^{k+1} \cap \Delta^{k+1} \in \cup_{i=0}^k \Phi_1^i$ ). By induction assumption, vertices in  $\Delta^{k+1} \setminus \mathcal{V}_1^k$  and vertices in  $\tilde{q}_*(\Delta^{k+1}) \setminus \mathcal{V}_2^k$  have the same cardinality, so we can choose an arbitrary bijection between them. Now we have  $q_*$  defined on  $\mathcal{V}_1^{k+1}$  and it remains to verify  $(*)$ . Given  $\Delta \in \mathcal{KS}(X(\Gamma_1))$ , let  $\{\Delta_i\}_{i=1}^d$  be the collection of elements in  $\cup_{i=0}^{k+1} \Phi_1^i$  such that  $\Delta_i \subset \Delta$ . Then  $\mathcal{V}_1^{k+1} \cap \Delta$  is the vertex set of  $\cup_{i=1}^d \Delta_i$ . Since  $\tilde{q}_*$  is a lattice isomorphism,  $\{\tilde{q}_*(\Delta_i)\}_{i=1}^d$  is exactly the collection of elements in  $\cup_{i=0}^{k+1} \Phi_2^i$  that are contained in  $\tilde{q}_*(\Delta)$ , and  $\mathcal{V}_2^{k+1} \cap \tilde{q}_*(\Delta)$  is the vertex set of  $\cup_{i=1}^d \tilde{q}_*(\Delta_i)$ , thus  $(*)$  is true for the  $k+1$  case by our construction.

It is clear from  $(*)$  that the map  $q_*$  defined as above is surjective. Pick distinct vertices  $v_1, v_2$  in  $\mathcal{P}(\Gamma_1)$ , if  $d(v_1, v_2) = 1$ , then by applying  $(*)$  to the maximal simplex contains  $v_1$  and  $v_2$ , we have  $d(q_*(v_1), q_*(v_2)) = 1$ . If  $d(v_1, v_2) = 2$ , pick vertex  $u$  such that  $d(v_1, u) = d(u, v_2) = 1$  and let  $\Delta_i$  be the maximal simplex containing  $v_i$  and  $u$  for  $i = 1, 2$ . We can apply  $(*)$  to  $\Delta_1, \Delta_2$  and  $\Delta_1 \cap \Delta_2$ , which would imply  $q_*(v_1) \neq q_*(v_2)$ . If  $d(v_1, v_2) > 2$ , we let  $\Delta_i$  be a maximal simplex containing  $v_i$  for  $i = 1, 2$ . Then  $\Delta_1 \cap \Delta_2 = \emptyset$ , thus  $\tilde{q}_*(\Delta_1) \cap \tilde{q}_*(\Delta_2) = \emptyset$  and  $q_*(v_1) \neq q_*(v_2)$  by  $(*)$ . Thus  $q_*$  is a bijection, moreover, pick vertices  $w_1, w_2 \in \mathcal{P}(\Gamma_2)$  with  $d(w_1, w_2) = 1$ , then  $d(q_*^{-1}(w_1), q_*^{-1}(w_2)) = 1$  by  $(*)$ . Now we can extent  $q_*$  to be a simplicial isomorphism between the 1-skeleton of  $\mathcal{P}(\Gamma_1)$  and the 1-skeleton of  $\mathcal{P}(\Gamma_2)$ . Since  $\mathcal{P}(\Gamma_1)$  and  $\mathcal{P}(\Gamma_2)$  are flag complexes, we can further extent  $q_*$  to obtain the required simplicial isomorphism.  $\square$

**Remark 2.21.**  $q_*$  defined as above actually satisfies

$$(2.22) \quad q_*(M) = \tilde{q}_*(M)$$

for any subcomplex  $M \in \mathcal{S}(\mathcal{P}(\Gamma_1))$ . If we only assume  $\text{Out}(G(\Gamma_1))$  is non-adjacent transvection free, by the same proof we still can get an injective simplicial map  $q_* : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$  such that

$$(2.23) \quad q_*(M) \subset \tilde{q}_*(M) \text{ and } M = q_*^{-1}(\tilde{q}_*(M))$$

for any subcomplex  $M \subset \mathcal{S}(\mathcal{P}(\Gamma_1))$ , but  $q_*$  may not be surjective.

The first inclusion in (2.23) is clear. To see the second equality, suppose there exists vertex  $v \notin M$  such that  $q_*(v) \in \tilde{q}_*(M)$  and let  $\Delta_v$  be the minimal element in  $\mathcal{KS}(\mathcal{P}(\Gamma_1))$  such that  $v \in \Delta_v$ . Note that  $\Delta_v$  is a simplex and  $\tilde{q}_*(\Delta_v)$  is the minimal element in  $\mathcal{KS}(\mathcal{P}(\Gamma_2))$  that contains  $q_*(v)$ , thus  $\tilde{q}_*(\Delta_v) \subset \tilde{q}_*(M)$ . But  $\tilde{q}_*$  is a lattice isomorphism, so we must have  $\Delta_v \subset M$ , which is a contradiction.

It is natural to ask to what extent is the converse of Theorem 2.20 true, namely, suppose  $s : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$  is a simplicial isomorphism, does  $s$  induce a map from  $G(\Gamma) \rightarrow G(\Gamma')$ ? Here is a natural construction. We identify  $G(\Gamma)$  and  $G(\Gamma')$  with the 0-skeleton of  $X(\Gamma)$  and  $X(\Gamma')$  respectively. Pick vertex  $p \in G(\Gamma)$ , let  $\{F_i\}_{i=1}^n$  be the collection of maximal standard flats containing  $p$ . For each  $i$ , let  $F'_i \subset X(\Gamma')$  be the unique maximal standard flat such that  $\Delta(F'_i) = s(\Delta(F_i))$ . One may wish to map  $p$  to  $\cap_{i=1}^n F'_i$ , which motivates the following definition:

**Definition 2.24.** The simplicial isomorphism  $s$  is *visible* if  $\cap_{i=1}^n F'_i \neq \emptyset$  for any  $p \in G(\Gamma)$ .

If  $G(\Gamma)$  has trivial centre (which means  $p = \cap_{i=1}^n F_i$ ) and  $s$  is visible, then it is easy to see  $s$  induces a unique map  $s_* : G(\Gamma) \rightarrow G(\Gamma')$ . If  $G(\Gamma)$  has non-trivial centre, then the map induced by  $s$  is not unique.

We have studied a sufficient condition for  $s$  to be visible in [Hua14a, Lemma 4.10]. In this paper, we will find a if and only if condition for the visibility of  $s$ .

### 3. THE STRUCTURE OF EXTENSION COMPLEX

Throughout this section, we identify  $\Gamma$  with the 1-skeleton of  $F(\Gamma)$ , and we will inexplicitly use Lemma 2.1 in various places.

**3.1. Tiers and branches of the extension complex.** Let  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  be the label-preserving simplicial map in Section 2.3.

Pick a standard geodesic  $l \subset X(\Gamma)$  and let  $\pi_l : X(\Gamma) \rightarrow l$  be the  $CAT(0)$  projection onto  $l$ . Suppose  $l_1 \subset X(\Gamma)$  is a standard geodesic such that  $d(\Delta(l_1), \Delta(l)) \geq 2$ . Then  $\pi_l(l_1)$  is a vertex in  $l$  by Lemma 2.11 and Lemma 2.12. Moreover,  $\pi_l(l_1) = \pi_l(l_2)$  if  $l_2$  is a standard geodesic parallel to  $l_1$  (see [Hua14a, Lemma 6.2]). Thus  $\pi_l$  induces a well-defined map  $\pi_{\Delta(l)}$  from the  $v(\mathcal{P}(\Gamma) \setminus St(\Delta(l)))$ , the set of vertices in  $\mathcal{P}(\Gamma) \setminus St(\Delta(l))$ , to  $v(l)$ .

**Lemma 3.1.** [Hua14a, Lemma 6.2] *If  $v_1$  and  $v_2$  are in the same connected component of  $\mathcal{P}(\Gamma) \setminus St(\Delta(l))$ , then  $\pi_{\Delta(l)}(v_1) = \pi_{\Delta(l)}(v_2)$ .*

Pick  $v \in \mathcal{P}(\Gamma)$ , and let  $l \subset X(\Gamma)$  be a standard geodesic such that  $\Delta(l) = v$ . Let  $\pi_{\Delta(l)} : v(\mathcal{P}(\Gamma) \setminus St(v)) \rightarrow v(l)$  be the map in Lemma 3.1. A  $v$ -tier is the full subcomplex spanned by  $\pi_{\Delta(l)}^{-1}(x)$ , here  $x$  is a vertex in  $l$  and  $x$  is called the *height* of the  $v$ -tier. A  $v$ -branch is the full subcomplex spanned by vertices in one connected component of  $\mathcal{P}(\Gamma) \setminus St(v)$ .

By the proof of Lemma 3.1, a  $v$ -branch has non-empty intersection with a  $v$ -tier if and only if it belongs to the  $v$ -tier, thus a  $v$ -tier consists of disjoint union of  $v$ -branches. Also note that a simplicial isomorphism  $s : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$  will map branches to branches, but it may not map tiers to tiers.

**Lemma 3.2.** *If the  $s$ -image of any  $v$ -tier of  $\mathcal{P}(\Gamma_1)$  is inside a  $s(v)$ -tier of  $\mathcal{P}(\Gamma_2)$ , then  $s$  is visible.*

*Proof.* Let  $p$ ,  $\{F_i\}_{i=1}^n$  and  $\{F'_i\}_{i=1}^n$  be as in Definition 2.24. By Lemma 2.2, it suffices to show  $F'_i \cap F'_j \neq \emptyset$  for any  $i \neq j$ . Suppose  $s$  is not visible. Then there exists a hyperplane  $h$  separating  $F'_i$  and  $F'_j$ . Let  $l'$  be a standard geodesic dual to  $h$  and let  $v' = \Delta(l')$ . Then the maximality of  $F'_i$  and  $F'_j$  implies there exist vertices  $v'_1 \in \Delta(F'_i)$  and  $v'_2 \in \Delta(F'_j)$  such that they are in different  $v'$ -tier. However,  $s^{-1}(v'_1)$  and  $s^{-1}(v'_2)$  are in the same  $s^{-1}(v')$ -tier, which yields a contradiction.  $\square$

Next we characterize  $v$ -branches in a  $v$ -tier for a certain class of  $\Gamma$ .

**Definition 3.3.** A graph  $\Gamma$  is of *type II* if  $\Gamma$  is connected and for every pair of distinct vertices  $v, w \in \Gamma$ ,  $\Gamma \setminus (lk(v) \cap lk(w))$  is connected.  $\Gamma$  is said to have *weak type II* if  $\Gamma$  is connected and for vertices  $v, w \in \Gamma$  such that  $d(v, w) = 2$ ,  $\Gamma \setminus (lk(v) \cap lk(w))$  is connected.  $G(\Gamma)$  or  $F(\Gamma)$  is of (weak) type II if  $\Gamma$  is of (weak) type II.

Note that  $\Gamma$  is connected if it is of (weak) type II.

Let  $K$  be a simplicial complex and let  $K_1, K_2$  be two subcomplex.  $K_1$  and  $K_2$  *contact* if there exist vertices  $v_i \in K_i$  for  $i = 1, 2$  such that  $v_1$  and  $v_2$  are adjacent.

Let  $v \in \mathcal{P}(\Gamma)$  be a vertex, and let  $l \in X(\Gamma)$  be a standard geodesic. Define  $P_v$  to be the parallel set  $P_l$  of  $l$ . Note that  $P_v$  does not depend on the choice of  $l$ . A subset  $K \subset P_v$  is *horizontal* if  $\pi_l(K)$  is a point ( $\pi_l : X(\Gamma) \rightarrow l$  is the  $CAT(0)$  projection) and  $\pi_l(K)$  is called the *height* of  $K$ .

Let  $\bar{v} \in \Gamma$  be the label of  $v$ . Then  $P_v$  is a standard subcomplex whose support (Definition 2.10) is  $St(\bar{v})$ .

**Lemma 3.4.** *Suppose  $\Gamma$  is of weak type II. Pick vertices  $v, w \in \mathcal{P}(\Gamma)$  such that  $d(v, w) = 2$ . Let  $u \in v^\perp \cap w^\perp$ . Then there exists vertex  $w'$  such that*

- (1)  $d(v, w') = 2$  and  $d(u, w') = 1$ .
- (2)  $w'$  and  $w$  are in the same  $v$ -branch.
- (3)  $P_v \cap P_{w'} \neq \emptyset$ .

*In particular, every  $v$ -branch contains a vertex  $w'$  such that  $P_{w'} \cap P_v \neq \emptyset$ .*

*Proof.* Let  $B$  be a  $v$ -branch containing  $w$ . Pick  $x \in P_w \cap P_u$  and let  $y \in P_v$  be the vertex such that  $d(x, y) = d(x, P_v)$ . The existence and uniqueness of such vertex follows from Lemma 2.4. Let us assume  $x \neq y$ , otherwise we are done by putting  $w' = w$ . Let  $\omega$  be a combinatorial geodesic connecting  $x$  and  $y$ . We claim  $\omega \subset P_u$ . By Lemma 2.3, it suffices to show  $y \in P_u$ . Let  $l_{x,u}$  be the standard geodesic with  $x \in l_{x,u}$  and  $\Delta(l_{x,u}) = u$ . Note that  $l_{x,u} \subset N_R(P_v)$  for some  $R > 0$ , then it follows from Lemma 2.11 and Lemma 2.4 that  $\mathcal{I}(l_{x,u}, P_v) = (l_{x,u}, l)$  for some standard geodesic  $l \subset P_v$ . Then  $y \in l$  and  $\Delta(l) = u$ , so  $y \in P_u$ .

Let  $\{x_i\}_{i=0}^n$  be vertices in  $\omega$  such that for  $0 \leq i \leq n-1$ ,  $[x_i, x_{i+1}]$  is a maximal sub-segment of  $\omega$  that is contained in a standard geodesic ( $x_0 = x$  and  $x_n = y$ ). Denote the standard geodesic containing  $[x_i, x_{i+1}]$  by  $l_i$  and let  $v_i = \Delta(l_i)$  for  $0 \leq i \leq n-1$ . Since  $\omega$  is a combinatorial geodesic, every dual hyperplane of some edge in  $\omega$  must separate  $x$  and  $P_v$ . Thus for each  $i$ , there exists a hyperplane dual to  $l_i$  which does not intersect  $P_v$ . It follows that

$$(3.5) \quad d(v_i, v) \geq 2$$

for all  $i$ . Since  $l_i \subset P_u$ , we also have

$$(3.6) \quad d(v_i, u) = 1.$$

Let  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  be the projection mentioned at the beginning of this section. Since  $l_{n-1} \cap P_v \neq \emptyset$ , it follows from (3.5) and (3.6) that

$$(3.7) \quad d(\pi(v_{n-1}), \pi(v)) = 2.$$

We claim  $v_0 \in B$ . Let  $K_{x_0} = (F(\Gamma))_{x_0}$  (see the paragraph before Theorem 2.20). First we show  $K_{x_0} \cap St(v)$  is contained in the intersection of the links of two vertices. Pick vertex  $s \in K_{x_0} \cap St(v)$ , and let  $l_s$  be the standard geodesic such that  $x_0 \in l_s$  and  $\Delta(l_s) = s$ . Let  $h$  be a hyperplane dual to  $l_{n-1}$  such that it separates  $x_0$  from  $P_v$ . Then  $h \cap l_s = \emptyset$  by (3.5), hence  $h$  separates  $l_s$  from  $P_v$ . It follows from Lemma 2.11 and Lemma 2.12 that  $\pi(s) \in (\pi(v_{n-1}))^\perp \cap (\pi(v))^\perp$ . Let  $K$  be the full subgraph spanned by  $(\pi(v_{n-1}))^\perp \cap (\pi(v))^\perp$ . Then  $t \notin St(v)$  for any vertex  $t \in K_{x_0}$  such that  $\pi(t) \notin K$ .

By (3.7),  $K$  does not separate  $\Gamma$ , so if  $\pi(w) \notin K$  and  $\pi(v_0) \notin K$ , then they can be connected by an edge path outside  $K$ , which lifts to a path in  $K_{x_0}$  connecting  $w$  and  $v_0$  outside  $St(v)$ , thus  $v_0 \in B$ . If  $\pi(w) \notin K$  and  $\pi(v_0) \in K$ , then we connected

$\pi(w)$  and  $\pi(v_{n-1})$  by an edge path outside  $K$ , then connect  $\pi(v_{n-1})$  and  $\pi(v_0)$  by an edge, this path also lifts to a path in  $K_{x_0}$  connecting  $w$  and  $v_0$  outside  $St(v)$ . The other cases can be dealt with in a similar way. We can repeat this process and argue inductively that actually  $v_i \in B$  for  $0 \leq i \leq n-1$ , then the lemma follows by take  $w' = v_{n-1}$ .  $\square$

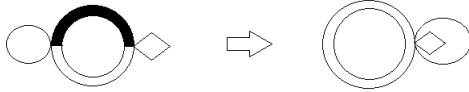
**Remark 3.8.** Let  $\Gamma$  and  $v$  be as above and pick vertex  $x \in X(\Gamma)$ . If there exists two vertices  $v_1, v_2 \in (F(\Gamma))_x$  such that they are in different  $v$ -branches, then  $v \in (F(\Gamma))_x$ . If this is not true, then  $x \notin P_v$ . In this case we pick a combinatorial geodesic of shortest length from  $x$  to a vertex in  $P_v$  and repeat the above argument to deduce that  $(F(\Gamma))_x \setminus St(v)$  is connected, which is a contradiction.

**Lemma 3.9.** *Suppose  $\Gamma$  is an arbitrary finite simplicial graph. Let  $v_1, v_2 \in \mathcal{P}(\Gamma) \setminus St(v)$  be two vertices such that  $P_{v_i} \cap P_v \neq \emptyset$  for  $i = 1, 2$ , and let  $\bar{v} = \pi(v)$ . If  $\pi(v_1)$  and  $\pi(v_2)$  are in different connected component of  $F(\Gamma) \setminus St(\bar{v})$ , then  $v_1$  and  $v_2$  are in different  $v$ -branch.*

*Proof.* We argue by contradiction and assume there exists a  $v$ -branch  $B$  which contains both  $v_1$  and  $v_2$ . For  $i = 1, 2$ , let  $l_i$  be a standard geodesic such that  $\Delta(l_i) = v_i$  and  $l_i \cap P_v \neq \emptyset$ . Note that  $l_1 \cap P_v$  and  $l_2 \cap P_v$  are of the same height. We claim there exist vertices  $x \in l_1 \setminus P_v$  and  $y \in l_2 \setminus P_v$  such that they can be connected by an edge path outside  $P_v$ .

Let  $\{w_i\}_{i=1}^n \subset \mathcal{P}(\Gamma) \setminus St(v)$  be vertices such that  $d(w_i, w_{i+1}) = 1$ ,  $w_1 = v_1$  and  $w_n = v_2$ . Pick  $x = x_1$  to be any vertices in  $l_1 \setminus P_v$ . For  $1 \leq i \leq n-1$ , let  $r'_i$  be a standard geodesic with  $\Delta(r'_i) = w_i$  and  $r'_i \cap P_{w_{i+1}} \neq \emptyset$  (set  $r'_n = l_2$ ). Let  $\omega_1$  be horizontal edge path in  $P_{w_1}$  connecting  $x_1$  and a vertex  $x_2 \in r'_1$ . Note that  $\omega_1 \cap P_v = \emptyset$  since  $P_v \cap P_{w_1}$  is either empty or horizontal in  $P_{w_1}$ . Let  $r_2$  be the standard geodesic such that  $x_2 \in r_2$  and  $\Delta(r_2) = w_2$ . If  $P_{w_2} \cap P_v = \emptyset$  or  $P_{w_2} \cap P_v$  and  $x_2$  have different height in  $P_{w_2}$ , let  $\omega_2$  be a horizontal edge path joining  $x_2$  and a vertex  $x_3 \in r'_2$ . If  $P_{w_2} \cap P_v$  and  $x_2$  have the same height in  $P_{w_2}$ , let  $\omega'_2$  be an edge in  $r_2$  joining  $x_2$  and another vertex  $x'_2$  and let  $\omega''_2$  be a horizontal edge path joining  $x'_2$  and a vertex  $x_3 \in r'_2$ . Set  $\omega_2 = \omega'_2 \cup \omega''_2$ , it clear that  $\omega_2 \cap P_v = \emptyset$  in both cases. We can define  $\omega_i$  and  $x_{i+1}$  for  $3 \leq i \leq n$  in the same way. Let  $y = x_{n+1}$  and the claim follows.

Let  $C_1$  be the component of  $F(\Gamma) \setminus St(\bar{v})$  that contains  $\pi(v_1)$  and  $C_2$  be the union of all other components. For  $i = 1, 2$ , let  $\Gamma_i$  be the full subgraph spanned by vertices in  $C_i \cup St(\bar{v})$ . Then  $St(\bar{v}) = \Gamma_1 \cap \Gamma_2$ . For  $i = 1, 2$ , let  $f_i : S(St(\bar{v})) \rightarrow S(\Gamma_i)$  be the natural embedding. Note that  $S(St(\bar{v})) \cong S(lk(\bar{v})) \times \mathbb{S}^1$ . Let  $h$  be an isometry of  $S(St(\bar{v}))$  which is identity on the  $S(lk(\bar{v}))$  factor and is a rotation of degree  $\pi$  on the  $\mathbb{S}^1$  factor. Define space  $\tilde{S}(\Gamma)$  by identifying  $S(St(\bar{v}))$  with  $S(\Gamma_1)$  via  $f_1$  and identifying  $S(St(\bar{v}))$  with  $S(\Gamma_2)$  via  $f_2 \circ h$ . Then there is a homotopy equivalence  $g : \tilde{S}(\Gamma) \rightarrow S(\Gamma)$  induced by collapsing the interval  $[e^{i0}, e^{i\pi}]$  in the  $\mathbb{S}^1$  factor of  $S(St(\bar{v}))$  to one point (see the following picture, where the black part is collapsed). It lifts to a cubical map  $\tilde{g} : \tilde{X}(\Gamma) \rightarrow X(\Gamma)$ .



Let  $M \subset P_v$  be the standard subcomplex such that  $l_1 \cap P_v \subset M$  and the support of  $M$  satisfies  $\Gamma_M = lk(\bar{v})$ . Then there exists a unique hyperplane  $\bar{h} \subset \tilde{X}(\Gamma)$



such that  $\tilde{g}(\bar{h}) = M$ . For  $i = 1, 2$ , let  $\bar{l}_i \subset \bar{X}(\Gamma)$  be the unique geodesic such that  $\tilde{g}(\bar{l}_i) = l_i$ . Then  $\bar{l}_1$  and  $\bar{l}_2$  have non-empty intersection with  $\tilde{g}^{-1}(P_v)$ . Since  $\tilde{g}^{-1}(P_v)$  is a lift of  $S(St(\bar{v}))$  in  $\bar{X}(\Gamma)$ , and  $\pi(\Delta(l_i)) \in C_i$  for  $i = 1, 2$ , we know that  $\bar{l}_1$  and  $\bar{l}_2$  are separated by  $\bar{h}$ . Let  $\omega = \cup_{i=1}^n \omega_i$  be the edge path connecting  $x$  and  $y$  in the previous paragraph. Then  $\tilde{g}^{-1}(\omega)$  is a compact connected subcomplex of  $X(\bar{\Gamma})$  with  $\tilde{g}^{-1}(\omega) \cap \bar{h} = \emptyset$ . Moreover,  $\tilde{g}^{-1}(\omega) \cap \bar{l}_i \neq \emptyset$  for  $i = 1, 2$ , which contradicts the separation property of  $\bar{h}$ .  $\square$

**Remark 3.10.** Let  $\Gamma$  be arbitrary. Let  $\omega \in X(\Gamma)$  be an edge path joining vertices  $x_1, x_2 \in P_v$ , and suppose  $\omega \setminus \{x_1, x_2\}$  stays inside one component of  $X(\Gamma) \setminus P_v$ . Then

- (1)  $x_1$  and  $x_2$  are of the same height in  $P_v$ .
- (2) For  $i = 1, 2$ , let  $e_i \subset \omega$  be the edge containing  $x_i$ , and let  $\bar{v}_i \in \Gamma$  be the label of  $e_i$ . Then  $\bar{v}_1$  and  $\bar{v}_2$  are in the same component of  $\Gamma \setminus St(\bar{v})$ .

(2) follows from the proof of Lemma 3.9.

Let  $v$  and  $\bar{v}$  be as in Lemma 3.9 and let  $C$  be a component of  $\Gamma \setminus St(\bar{v})$ . We define  $\partial C$  to be the full subgraph spanned by vertices in  $\bar{C} \setminus C$ , here  $\bar{C}$  is the closure of  $C$ . Equivalently,  $\partial C$  is the full subgraph spanned by vertices in  $\{u \in \Gamma \setminus C \mid \text{there exists vertex } w \in C \text{ such that } d(w, u) = 1\}$ . Similarly, for every  $v$ -branch  $B \subset \mathcal{P}(\Gamma)$ , we define the *boundary* of  $B$ , denoted by  $\partial B$ , to be the full subcomplex spanned by vertices in  $\{u \in \mathcal{P}(\Gamma) \setminus B \mid \text{there exists vertex } w \in B \text{ such that } d(w, u) = 1\} = \{u \in St(v) \mid \text{there exists vertex } w \in B \text{ such that } d(w, u) = 1\}$ . Such  $\partial B$  is called a *v-peripheral subcomplex* of  $\mathcal{P}(\Gamma)$ . We caution the reader that  $B \cup \partial B$  may not equal to the closure of  $B$ .

A subcomplex  $K \subset P_v$  is called a *v-peripheral subcomplex (of type  $\partial C$ )* if  $K$  is a standard subcomplex and  $\Gamma_K = \partial C$  for some component  $C$  of  $\Gamma \setminus St(\bar{v})$ . If the vertex set of  $\partial C$  is properly contained in  $\bar{v}^\perp$ , then there are infinitely many  $v$ -peripheral subcomplexes of type  $\partial C$  which are of the same height.

**Lemma 3.11.** *Let  $\Gamma$  be arbitrary. Let  $x_1, x_2, \bar{v}_1, \bar{v}_2$  and  $P_v$  be as in Remark 3.10 and let  $C$  be the component of  $\Gamma \setminus St(\bar{v})$  containing  $\bar{v}_1$  and  $\bar{v}_2$ . Then  $x_1$  and  $x_2$  are in the same  $v$ -peripheral subcomplex of type  $\partial C$ .*

*Proof.* For  $i = 1, 2$ , let  $K_i$  be the  $v$ -peripheral subcomplex of type  $\partial C$  such that  $x_i \in K_i$ . Note that  $K_1$  and  $K_2$  are horizontal subcomplex of  $\mathcal{P}(\Gamma)$  of the same height. We argue by contradiction and suppose  $K_1 \neq K_2$ . Then  $K_1 \cap K_2 = \emptyset$ . We claim there exists an edge  $e \in P_v$  such that its label  $v_e$  does not belong to  $\partial C$  and the hyperplane dual to  $e$  separates  $K_1$  from  $K_2$ . To see this, pick vertices  $x \in K_1$  and  $y \in K_2$  such that  $d(x, y) = d(K_1, K_2)$ . Let  $\omega_1$  be a combinatorial geodesic joining  $x$  and  $y$ . Then  $\omega_1 \subset P_v$  by Lemma 2.3. Moreover, every hyperplane dual to some edge in  $\omega_1$  separates  $K_1$  and  $K_2$ . Thus there exists an edge  $e \in \omega_1$  such that  $v_e \notin \partial C$ , otherwise we would have  $\omega_1 \subset K_1$ .

Let  $h_e$  be the hyperplane dual to  $e$  and let  $N_{h_e}$  be the carrier of  $h_e$ . Then  $h_e$  separates  $x_1, x_2$  and there exists an edge  $e' \subset \omega$  parallel to  $e$  ( $\omega$  is the path in Remark 3.10). Pick endpoint  $y \in e'$  and let  $\omega_2 \subset N_{h_e}$  be an edge path of shortest combinatorial length connecting  $y$  and  $P_v \cap N_{h_e}$ . Let  $x_3$  be the other endpoint of  $\omega_2$  and let  $e'' \subset \omega_2$  be the edge containing  $x_3$ . Then  $d(v_{e''}, v_e) = 1$  ( $v_{e''}$  is the label of  $e''$ ), and it follows from  $v_e \notin \partial C$  that

$$(3.12) \quad v_{e''} \notin C.$$

Let  $\omega_2$  be an edge path connecting  $x_1$  and  $x_3$  obtained by first following  $\omega$  from  $x_1$  to  $y$ , then following  $\omega_2$  until  $x_3$ . Then applying Remark 3.10 to  $\omega_2$  yields a contradiction to (3.12).  $\square$

**Corollary 3.13.** *Suppose  $\Gamma$  is connected. Let  $v$  and  $\bar{v}$  be as in Lemma 3.9 and pick a component  $C$  of  $\Gamma \setminus St(\bar{v})$ . Suppose  $K_1$  and  $K_2$  are two distinct  $v$ -peripheral subcomplexes of type  $\partial C$  and they have the same height. Let  $w_1, w_2 \in \mathcal{P}(\Gamma) \setminus St(v)$  be vertices such that  $\pi(w_i) \in C$  for  $i = 1, 2$ . Suppose  $P_{w_i} \cap K_i \neq \emptyset$  for  $i = 1, 2$ . Then  $w_1$  and  $w_2$  are in different  $v$ -branches.*

*Proof.* For  $i = 1, 2$ , let  $l_i$  be a standard geodesic such that  $l_i \cap K_i \neq \emptyset$  and  $\Delta(l_i) = w_i$ . If  $w_1$  and  $w_2$  are in the same  $v$ -branch, then the argument in the second paragraph of the proof of Lemma 3.9 implies there exists an edge path  $\omega \subset X(\Gamma) \setminus P_v$  connecting a vertex in  $l_1 \setminus P_v$  to a vertex in  $l_2 \setminus P_v$ . Then it follows from Lemma 3.11 that  $K_1 = K_2$ , which is a contradiction.  $\square$

**Corollary 3.14.** *Suppose  $\Gamma$  is of type II. Let  $v$  and  $\bar{v}$  be as before, and let  $l \subset X(\Gamma)$  be a standard geodesic such that  $\Delta(l) = v$ . Pick vertex  $x \in l$ , then*

- (1) *There is a 1-1 correspondence between  $v$ -branches in the  $v$ -tier of height  $x$  and pairs  $(C, K)$ , where  $C$  is a component in  $\Gamma \setminus St(\bar{v})$  and  $K$  is a  $v$ -peripheral subcomplexes in  $X(\Gamma)$  of height  $x$  such that  $\Gamma_K = \partial C$ . Moreover, let  $B$  be the  $v$ -branch corresponding to  $(C, K)$ . Then  $\partial B = \Delta(K)$ .*
- (2) *For every  $v$ -peripheral subcomplex  $A \subset \mathcal{P}(\Gamma)$ , there exists a unique  $v$ -peripheral subcomplex  $K \subset X(\Gamma)$  of height  $x$  such that  $\Delta(K) = A$ .*
- (3) *Let  $A$  be as above. Then there are only finite many  $v$ -branches with boundary equal to  $A$  in a  $v$ -tier.*
- (4) *Let  $v_1, v_2 \in \mathcal{P}(\Gamma)$  be two different vertices and  $B_i \subset \mathcal{P}(\Gamma)$  be a  $v_i$ -branch for  $i = 1, 2$ . Then  $B_1 \neq B_2$ .*
- (5) *Let  $v_1, v_2$  be as above. Then  $\mathcal{P}(\Gamma) \setminus (lk(v_1) \cap lk(v_2))$  is connected.*

*Proof.* For  $i = 1, 2$ , pick pairs  $(C_i, K_i)$  as above, let  $w_i \in \mathcal{P}(\Gamma)$  be a vertex such that  $\pi(w_i) \in C_i$  and  $P_{w_i} \cap K_i \neq \emptyset$ , we claim  $w_1$  and  $w_2$  are in the same  $v$ -branch if and only if  $C_1 = C_2$  and  $K_1 = K_2$ . Assuming the claim, then the first part of (1) follows from Lemma 3.4. The only if direction follows from Lemma 3.9 and Corollary 3.13. For the other direction, pick vertex  $x_i \in P_{w_i} \cap K_i$ , it suffices to consider the case when  $x_1$  and  $x_2$  are joined by an edge  $e \subset K_1$ . Let  $l_e$  be the standard geodesic containing  $e$  and let  $v_e = \Delta(l_e)$ . Then  $\pi(v_e) \in \partial C_1$  and there exists  $\bar{u} \in C_1$  such that

$$(3.15) \quad d(\bar{u}, \pi(v_e)) = 1.$$

For  $i = 1, 2$ , let  $\bar{\omega}_i \subset C_1$  be the edge path connecting  $\bar{u}$  and  $\pi(w_i)$ . Then we lift  $\bar{\omega}_i$  to an edge path  $\omega_i \subset (F(\Gamma))_{x_i}$ . (3.15) implies we can concatenate  $\omega_1$  and  $\omega_2$  to obtain a path connecting  $w_1$  and  $w_2$  outside  $St(v)$ .

Now we prove the second statement of (1). Pick pair  $(C, K)$  as above and  $B$  be the associated  $v$ -branch. Since  $\Gamma_K = \partial C$ , for each standard geodesic  $l \subset K$ , there exists a standard geodesic  $l'$  such that (1)  $\pi(\Delta(l')) \in C$ ; (2)  $l'$  and  $l$  span a 2-flat. Thus  $\Delta(l') \in B$  and  $\Delta(l) \in \partial B$ . Hence  $\Delta(K) \subset \partial B$ . Now we prove the other direction. Pick  $u \in \partial B$ . By Lemma 3.4, we can assume there exists  $w' \in B$  such that  $d(w', u) = 1$  and  $P_{w'} \cap P_v \neq \emptyset$ . Then  $\pi(w') \in C$  by Lemma 3.9, hence  $\pi(u) \in \partial C$ . Note that  $P_{w'} \cap P_u \neq \emptyset$  and  $P_v \cap P_u \neq \emptyset$ . Thus  $P_v \cap P_u \cap P_{w'} \neq \emptyset$  by Lemma 2.2. Pick vertex  $z$  in this triple intersection. Then  $z \in K$  by Lemma 3.13. Let  $l_z$

be a standard geodesic such that  $z \in l_z$  and  $\Delta(l_z) = u$ . Since  $\pi(u) \in \partial C = \Gamma_K$ ,  $l_z \subset K$ . Thus  $u \in \Delta(K)$ .

The existence in (2) follows from Lemma 3.4 and the above discussion. Let  $K_1$  and  $K_2$  be two  $v$ -peripheral subcomplexes of the same height such that  $\Delta(K_1) = \Delta(K_2) = A$ . Then the Hausdorff distance  $d_H(K_1, K_2) < \infty$  by Lemma 2.11. If  $K_1 \cap K_2 \neq \emptyset$ , then  $K_1 = K_2$  since  $\Gamma_{K_1} = \Gamma_{K_2}$ . Otherwise there exists a horizontal edge  $e \subset P_v$  such that the hyperplane dual to  $e$  separates  $K_1$  from  $K_2$  (note that  $K_1$  and  $K_2$  are horizontal). Suppose  $\bar{w} \in \Gamma$  is the label of  $e$ . Then  $d(\bar{w}, \bar{v}) = 1$  and  $\Gamma_{K_1} \subset St(\bar{w}) \setminus \{\bar{w}\}$  by Lemma 2.11. It follows that  $lk(\bar{w}) \cap lk(\bar{v})$  contains  $\Gamma_{K_1}$ , hence separates  $\Gamma$ , which is a contradiction. (3) follows from (1) and (2).

To see (4), suppose  $B_1 = B_2$ . By (1), there exist standard subcomplexes  $K_i \subset P_{v_i}$  for  $i = 1, 2$  such that  $\Delta(K_i) = \partial B_i$ . Then  $K_1$  and  $K_2$  are parallel, hence  $\Gamma_{K_1} = \Gamma_{K_2}$ . Let  $\bar{v}_i = \pi(v_i)$  for  $i = 1, 2$ . Then  $\Gamma_{K_1} \subset lk(\bar{v}_1) \cap lk(\bar{v}_2)$ . By Lemma 3.4, there exists vertex  $w \in B_1$  such that  $\bar{w} = \pi(w) \in C$  where  $C$  is a component of  $\Gamma \setminus St(v_1)$  with  $\partial C = \Gamma_{K_1}$ . Therefore,  $\Gamma_{K_1}$  separates  $\bar{v}_1$  from  $\bar{w}$ , so does  $lk(\bar{v}_1) \cap lk(\bar{v}_2)$ . This leads to a contradiction in the case  $\bar{v}_1 \neq \bar{v}_2$ . Suppose  $\bar{v}_1 = \bar{v}_2$ . Then  $P_{v_1}$  and  $P_{v_2}$  are standard complexes with the same support. Thus  $P_{v_1} \cap P_{v_2} = \emptyset$ , otherwise we would have  $P_{v_1} = P_{v_2}$  and  $v_1 = v_2$ . Let  $h$  be a hyperplane separating  $P_{v_1}$  and  $P_{v_2}$  such that the carrier of  $h$  intersects  $P_{v_1}$ . Then the label of edges dual to  $h$ , denoted by  $\bar{v}_h$ , satisfies  $d(\bar{v}_h, \bar{v}_1) \geq 2$ . It follows from Lemma 2.12 that  $\Gamma_{K_1} \subset lk(\bar{v}_h)$ . Thus  $lk(\bar{v}_1) \cap lk(\bar{v}_h)$  separates  $\Gamma$ , which is a contradiction.

To see (5), first we assume  $d(\bar{v}_1, \bar{v}_2) \neq 0$ . Since  $\Gamma$  is of type II, for any component  $C$  of  $\Gamma \setminus St(\bar{v}_1)$ ,

$$(3.16) \quad \partial C \setminus (lk(\bar{v}_1) \cap lk(\bar{v}_2)) \neq \emptyset.$$

Let  $B$  be a  $v_1$ -branch. Then (1) and (2) imply there exist standard subcomplex  $K \subset P_{v_1}$  and component  $C'$  of  $\Gamma \setminus St(\bar{v}_1)$  such that  $\partial B = \Delta(K)$  and  $\Gamma_K = \partial C'$ . Hence  $\partial C' = \pi(\partial B) \cap \Gamma$  (recall that we have identified  $\Gamma$  with the 1-skeleton of  $F(\Gamma)$ ). It follows from (3.16) that  $\partial B \setminus (lk(v_1) \cap lk(v_2)) \neq \emptyset$ , otherwise we would have  $\partial C' \subset \pi(\partial B) \subset \pi(lk(v_1) \cap lk(v_2)) \subset lk(\bar{v}_1) \cap lk(\bar{v}_2)$ . Then every vertex in  $B$  can be connected to  $v_1$  outside  $lk(v_1) \cap lk(v_2)$  and (5) follows.

If  $\bar{v}_1 = \bar{v}_2$ , then  $P_{v_1} \cap P_{v_2} = \emptyset$ . Hence  $d(v_1, v_2) \geq 2$  and  $lk(v_1) \cap lk(v_2) = St(v_1) \cap St(v_2)$ . Let  $h$  and  $\bar{h}_v$  be as in the proof of (4). Let  $l_h$  be a standard geodesic dual to  $l$  and let  $v_h = \Delta(l_h)$ . Note that  $d(v_h, v_1) \geq 2$  and  $\pi(v_h) = \bar{v}_h \neq \bar{v}_1$ . It suffices to prove  $St(v_1) \cap St(v_2) \subset St(v_1) \cap St(v_h)$ , since this reduces the current case to the previous case. Let  $k$  be a vertex in  $St(v_1) \cap St(v_2)$ . Then  $P_k$  has nontrivial intersection with both  $P_{v_1}$  and  $P_{v_2}$ . Since  $h$  separates  $P_{v_1}$  and  $P_{v_2}$ ,  $P_k \cap h \neq \emptyset$ . Hence  $l_h \subset P_k$  and  $d(k, v_h) \leq 1$ .  $\square$

### 3.2. Quasi-isometry invariance of type II and weak type II.

**Lemma 3.17.** *If  $\Gamma$  is of weak type II, then*

- (1) *There is no non-adjacent transvection in  $\text{Out}(G(\Gamma))$ .*
- (2)  *$\mathcal{P}(\Gamma)$  is of weak type II.*

*Proof.* (1) follows directly from the definition. To see (2), pick distinct vertices  $v_1, v_2 \in \mathcal{P}(\Gamma)$  such that  $d(v_1, v_2) = 2$ . For  $i = 1, 2$ , let  $\bar{v}_i = \pi(v_i)$ . The case  $d(\bar{v}_1, \bar{v}_2) = 2$  and  $\bar{v}_1 = \bar{v}_2$  has been dealt with in Corollary 3.14 (5). Now we assume  $d(\bar{v}_1, \bar{v}_2) = 1$ . If  $P_{v_1} \cap P_{v_2} \neq \emptyset$ , then it is a standard complex whose support is the intersection of the supports of  $P_{v_1}$  and  $P_{v_2}$ , which is  $St(\bar{v}_1) \cap St(\bar{v}_2)$ . Thus

$d(v_1, v_2) = 1$ , which yields a contradiction. So  $P_{v_1} \cap P_{v_2} = \emptyset$  and we have reduced to second case of Corollary 3.14 (5).  $\square$

**Lemma 3.18.** [Hua14a, Lemma 5.1] *Let  $\Gamma$  be a finite simplicial graph. Pick a vertex  $\bar{w} \in \Gamma$  and let  $\Gamma_{\bar{w}}$  be the minimal stable subgraph containing  $\bar{w}$ . Denote  $\Gamma_1 = lk(\bar{w})$  and  $\Gamma_2 = lk(\Gamma_1)$  (see Section 2.1 for definition of links), then either of the following is true:*

- (1)  $\Gamma_{\bar{w}}$  is a clique. In this case  $St(\bar{w})$  is a stable subgraph.
- (2) Both  $\Gamma_1$  and the join  $\Gamma_1 \circ \Gamma_2$  of  $\Gamma_1$  and  $\Gamma_2$  are stable subgraphs of  $\Gamma$ . Moreover,  $\Gamma_2$  is disconnected.

**Lemma 3.19.** *Suppose  $\mathcal{P}(\Gamma)$  is of weak type II. Let  $q : G(\Gamma) \rightarrow G(\Gamma')$  be a quasi-isometry. Then  $q$  induces a simplicial isomorphism  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$ .*

The proof is a variation of [Hua14a, Theorem 5.3].

*Proof.* Let  $\Gamma = \Gamma_1 \circ \Gamma_2$  be any join decomposition. Then  $\Gamma$  is of weak type II if and only if each  $\Gamma_i$  is of weak type II. So in the light of Theorem 2.16, we only need to focus on the case when  $\Gamma$  is irreducible and is not a clique. In this case  $\Gamma'$  is also irreducible and is not a clique, hence  $diam(\mathcal{P}(\Gamma)) = \infty$  and  $diam(\mathcal{P}(\Gamma')) = \infty$  [KK13, Lemma 26 (5)].

Let  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$  be the simplicial embedding in Remark 2.21. Suppose  $q_*$  is not surjective. Then there exists a vertex  $w' \in \mathcal{P}(\Gamma')$  which is not in the image of  $q_*$ . Let  $\bar{w}' = \pi(w')$  and let  $l'$  be a standard geodesic  $\Delta(l') = w'$ . We apply Lemma 3.18 to  $\bar{w}' \in \Gamma'$ . If case (1) is true, let  $F'$  be the standard flat in  $X(\Gamma')$  such that  $l' \subset F'$  and  $\Gamma_{F'} = \Gamma_{w'}$ . Since  $\Gamma_{w'}$  is stable,  $w' \in \Delta(F') \subset q_*(\mathcal{P}(\Gamma))$ , which is a contradiction.

If case (2) is true, let  $\Gamma'_1 = lk(\bar{w}')$  and  $\Gamma'_2 = lk(\Gamma'_1)$ . Take  $K'_1$  and  $K'$  to be the standard subcomplexes in  $X(\Gamma')$  such that (1)  $\Gamma_{K'_1} = \Gamma'_1$  and  $\Gamma_{K'} = \Gamma'_1 \circ \Gamma'_2$ ; (2)  $l' \subset K'$  and  $K'_1 \subset K'$ . Set  $M'_1 = \Delta(K'_1)$  and  $M' = \Delta(K')$ . Let  $K'_2$  be an orthogonal complement of  $K'_1$  in  $K'$ , i.e.  $K'_2$  is a standard subcomplex such that  $\Gamma_{K'_2} = \Gamma'_2$  and  $K' = K'_1 \times K'_2$ . It follows that  $M'$  has a join decomposition  $M' = M'_1 * M'_2$  for  $M'_2 = \Delta(K'_2)$ . By construction,  $w' \in M'$  and  $lk(w') = M'_1$ .

Since  $K'$  and  $K'_1$  are stable, there exist stable standard subcomplexes  $K$  and  $K_1$  in  $X(\Gamma)$  such that the Hausdorff distances satisfy  $d_H(q(K), K') < \infty$  and  $d_H(q(K_1), K'_1) < \infty$ . Moreover, by applying Theorem 2.16 to the quasi-isometry between  $K$  and  $K'$ , there exists standard subcomplex  $K_2 \in K$  such that  $K = K_1 \times K_2$  and  $K_2$  is quasi-isometric to  $K'_2$ , thus  $\Gamma_{K_2}$  is also disconnected. Let  $M_i = \Delta(K_i) \subset \mathcal{P}(\Gamma)$  for  $i = 1, 2$  and  $M = M_1 * M_2 = \Delta(K)$ . Then  $q_*^{-1}(M'_1) = M_1$  by (2.22) and (2.23). Since  $lk(w') = M'_1$  and  $w' \notin q_*(\mathcal{P}(\Gamma))$ ,

$$(3.20) \quad q_*^{-1}(St(w')) = M_1.$$

Let  $I = q_*(\mathcal{P}(\Gamma))$ . Then  $I$  is  $D$ -dense in  $\mathcal{P}(\Gamma')$  for some constant  $D > 0$ . To see this, it suffices to show  $\Gamma'$  contains a stable clique, but this follows from the existence of stable clique in  $\Gamma$ .

We claim every  $w'$ -tier contains vertices arbitrarily far from  $w'$ . To see this, let  $l'$  be a standard geodesic such that  $\Delta(l') = w'$ . We consider the action  $G(\Gamma) \curvearrowright X(\Gamma)$  by deck transformations and the induced action  $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$ . Then the stabilizer of  $l'$  is isomorphic to  $\mathbb{Z}$ . Moreover, this copy of  $\mathbb{Z}$  acts transitively on the collection of  $w'$ -tires. Now the claim follows from  $diam(\mathcal{P}(\Gamma')) = \infty$ .

We pick vertices  $w'_1, w'_2 \in \mathcal{P}(\Gamma')$  such that they are not in the same  $w'$ -tier and  $d(w'_i, w') > D+5$  for  $i = 1, 2$ . Let  $u'_i \in I$  be a vertex such that  $d(u'_i, w'_i) \leq D$ . Then  $u'_1$  and  $u'_2$  is separated by  $St(w')$  and  $d(u'_i, I \cap St(w')) > 4$ . Define  $u_i = q_*^{-1}(u'_i)$ , then  $u_1$  and  $u_2$  are in different components of  $\mathcal{P}(\Gamma) \setminus q_*^{-1}(St(w')) = \mathcal{P}(\Gamma) \setminus M_1$ , and

$$(3.21) \quad d(u_i, M_1) > 4.$$

Since  $\Gamma_{K_2}$  is disconnected, there exists vertex  $v_1, v_2 \in M_2$  such that  $d(v_1, v_2) = 2$ . Recall that  $M = M_1 * M_2 \subset \mathcal{P}(\Gamma)$ , so  $M_1 \subset lk(v_1) \cap lk(v_2)$ . Moreover,  $d(u_i, lk(v_1) \cap lk(v_2)) > 0$  by (3.21), so  $u_1$  and  $u_2$  are separated by  $lk(v_1) \cap lk(v_2)$ , which is contradictory to our assumption on  $\mathcal{P}(\Gamma)$ . So  $q_*$  must be surjective.  $\square$

**Lemma 3.22.** *If  $\mathcal{P}(\Gamma)$  is of weak type II, then  $\Gamma$  is of weak type II.*

*Proof.* Suppose  $\Gamma$  is not of weak type II. Then there exist vertices  $\bar{v}_1, \bar{v}_2 \in \Gamma$  such that  $d(\bar{v}_1, \bar{v}_2) = 2$  and  $\Gamma \setminus lk(\bar{v}_1) \cap lk(\bar{v}_2)$  is disconnected. Then we can find component  $C$  of  $\Gamma \setminus St(\bar{v}_1)$  such that  $\partial C \subset lk(\bar{v}_1) \cap lk(\bar{v}_2)$ . Pick vertex  $x_0 \in P_{v_1}$  and let  $K \subset P_{v_1}$  be the standard subcomplex with support  $= \partial C$  that contains  $x_0$ . Pick vertex  $\bar{v}_3 \in C$  (it is possible that  $\bar{v}_3 = \bar{v}_2$ ). For  $i = 1, 2, 3$ , let  $v_i \in (F(\Gamma))_{x_0}$  be the lift of  $\bar{v}_i$ . Then  $\Delta(K) \subset St(v_1) \cap St(v_2) = lk(v_1) \cap lk(v_2)$ . Let  $B$  be the  $v_1$ -branch that contains  $v_3$ . We claim  $\partial B = \Delta(K)$ , which then implies  $v_1$  and  $B$  are in different components of  $\mathcal{P}(\Gamma) \setminus (lk(v_1) \cap lk(v_2))$ .

$\Delta(K) \subset \partial B$  follows from the argument in (1) of Corollary 3.14. To see the other direction, pick  $w_1 \in \partial B$  and  $w_2 \in B$  such that  $d(w_1, w_2) = 1$ . Let  $l_3$  be the geodesic such that  $x_0 \in l_3$  and  $\Delta(l_3) = v_3$ . If  $P_{w_2} \cap P_{v_1} = \emptyset$ , then by the argument in Lemma 3.9, we can find an edge path  $\omega \in X(\Gamma) \setminus P_{v_1}$  connecting  $x \in l_3 \setminus \{x_0\}$  and  $y \in P_{w_2} \cap P_{w_1}$ . Let  $\omega_1 \subset P_{w_1}$  be a horizontal edge path connecting  $y$  and some vertex  $z \in P_{w_1} \cap P_{v_1}$ . Such path exists since  $P_{w_1} \cap P_{v_1}$  contains a standard geodesic  $l$  with  $\Delta(l) = w_1$ . We can also assume  $\omega_1 \cap P_{v_1} = \{z\}$ . Let  $e \subset \omega_1$  be the edge containing  $z$  and  $\bar{v}_e$  be the label of  $e$ . Since  $\omega_1$  is horizontal in  $P_{w_1}$ ,

$$(3.23) \quad d(\bar{v}_e, \pi(w_1)) = 1.$$

Let  $\omega'$  be the edge path obtained by (1) going from  $x_0$  to  $x$  along  $l_3$ ; (2) going from  $x$  to  $y$  along  $\omega$ ; (3) going from  $y$  to  $z$  along  $\omega_1$ . By applying Remark 3.10 and Lemma 3.11 to  $\omega'$ , we have  $\bar{v}_e \in C$  and  $z \in K$ . Hence  $\pi(w_1) \in \partial C$  by (3.23). This together with  $z \in P_{w_1}$  and  $z \in K$  imply  $w_1 \in \Delta(K)$ . If  $P_{w_2} \cap P_{v_1} \neq \emptyset$ , we still have  $w_1 \in \Delta(K)$  by the proof of the second statement of Corollary 3.14 (1). Thus  $\partial B \subset \Delta(K)$ .  $\square$

Actually the above argument also shows that if  $\mathcal{P}(\Gamma)$  is of type II, then  $\Gamma$  is of type II. The following corollary follows from (5) of Corollary 3.14, Lemma 3.17, Lemma 3.19 and Lemma 3.22.

**Corollary 3.24.**  *$\Gamma$  is of (weak) type II if and only if  $\mathcal{P}(\Gamma)$  is of (weak) type II. If  $G(\Gamma')$  is quasi-isometric to  $G(\Gamma)$ , then  $\Gamma'$  is also of (weak) type II.*

For any  $v$ -branch  $B$ , we denote the full subcomplex spanned by vertices in  $B$  and  $\partial B$  by  $\bar{B}$ . For any component  $L$  of  $X(\Gamma) \setminus P_v$ , we use  $\partial L$  to denote the full subcomplex spanned by vertices outside  $L$  which are adjacent to some vertex in  $L$ , and use  $\bar{L}$  to denote the full subcomplex spanned by vertices in  $L$  and  $\partial L$ . Note that  $\bar{B}$  may not be the closure of  $B$  and  $\bar{L}$  may not be the closure of  $L$ .

For any subcomplex  $K \subset X(\Gamma)$ , let  $\{F_\lambda\}_{\lambda \in \Lambda}$  be the collection of standard flats in  $K$  and define  $\Delta(K) = \cup_{\lambda \in \Lambda} \Delta(F_\lambda)$ .

**Lemma 3.25.** *If  $K$  is a convex subcomplex, then  $\Delta(K)$  is a full subcomplex of  $\mathcal{P}(\Gamma)$ .*

*Proof.* Let  $F \subset X(\Gamma)$  be a standard flat such that vertices of  $\Delta(F)$  are in  $\Delta(K)$ . Suppose  $(F', K') = \mathcal{I}(F, K)$ . Lemma 2.4 (4) implies that every standard geodesic of  $F$  is contained in a  $R$ -neighbourhood of  $F'$  for some  $R > 0$ . However,  $F'$  is a convex subcomplex of  $F$ , so actually  $F = F'$ . Hence  $\Delta(F) \subset \Delta(K)$ .  $\square$

**Lemma 3.26.** *Pick vertex  $v \in \mathcal{P}(\Gamma)$  and let  $\bar{v} = \pi(v)$ . Let  $L$  be a component of  $X(\Gamma) \setminus P_v$ . Then*

- (1)  $\partial L$  is a  $v$ -peripheral subcomplex of  $X(\Gamma)$ . Moreover, the topological boundary  $\partial^{Top} L$  of  $L$  is contained in  $\partial L$ , and contains the 1-skeleton of  $\partial L$ .
- (2)  $\bar{L} = L \cup \partial L$ , and  $L$  is a convex subcomplex of  $X(\Gamma)$ .

*If  $\Gamma$  is of weak type II and  $q : X(\Gamma) \rightarrow X(\Gamma')$  is a quasi-isometry, then*

- (3) *There is a 1-1 correspondence between  $v$ -branches and components of  $X(\Gamma) \setminus P_v$ . In particular, there exists a unique  $v$ -branch  $B$  such that  $\Delta(\bar{L}) = \bar{B}$  and  $\Delta(\partial L) = \partial B$ .*
- (4) *There is a component  $L'$  of  $X(\Gamma') \setminus P_{q_*(v)}$  such that  $d_H(q(L), L') < \infty$ .*
- (5) *For any component  $C$  of  $\Gamma \setminus St(\bar{v})$ ,  $\partial C$  is a stable subgraph of  $\Gamma$ .*

*Proof.* To see (1), note that Remark 3.10 implies there exists a component  $C$  of  $\Gamma \setminus St(\bar{v})$  such that the label of any edge which connects a vertex in  $L$  and a vertex outside  $L$  is inside  $C$ . Pick a vertex in  $X(\Gamma) \setminus L$  which is adjacent to some vertex in  $L$ , and let  $K$  be the  $v$ -peripheral subcomplex of type  $\partial C$  that contains this vertex. Then Lemma 3.11 implies vertex set of  $\partial L$  is contained in  $K$ , hence  $\partial L \subset K$ . Note that  $\partial^{Top} L$  is a subcomplex whose vertex set is the same as  $\partial L$ , hence  $\partial^{Top} L \subset L$ . On the other hand, the proof of the first statement of Corollary 3.14 (1) implies every edge of  $K$  is contained in  $\partial^{Top} L$ . Thus (1) follows.

To see (2), note that  $L \cup \partial L$  is a subcomplex, moreover, if a collection of mutual orthogonal edges emanating from the same vertex is contained in  $L \cup \partial L$ , then the cube spanned by these edges is contained in  $L \cup \partial L$ . Thus  $L \cup \partial L$  is a convex subcomplex, in particular it is a full subcomplex, hence (2) follows.

Now we prove (3). For  $i = 1, 2$ , let  $e_i \subset X(\Gamma)$  be an edge such that one of its endpoint  $x_{i1} \in X(\Gamma) \setminus P_v$  and another endpoint  $x_{i2} \in P_v$ . Let  $\bar{v}_i \in \Gamma$  be the label of  $e_i$ , and let  $C_i$  be the component of  $\Gamma \setminus St(\bar{v})$  that contains  $\bar{v}_i$ . We claim  $x_{11}$  and  $x_{21}$  are in the same component of  $X(\Gamma) \setminus P_v$  if and only if  $C_1 = C_2$  and  $x_{21}$  and  $x_{22}$  belong to the same  $v$ -peripheral subcomplex of  $\partial C_1$ . Then we have a 1-1 correspondence between components of  $X(\Gamma) \setminus P_v$  and the pair  $(C, K)$  as in (1) of Corollary 3.14 and the first part of (3) follows. The only if part of the claim follows from Remark 3.10 and Lemma 3.11. Note that  $C_1$  contains more than one point (otherwise  $\Gamma$  will not be of weak type II), so the if direction holds in the special case when  $\bar{v}_1 = \bar{v}_2$ ,  $x_{21} = x_{22}$  and  $x_{11} \neq x_{12}$ . The general case follows from the argument in (1) of Corollary 3.14.

Suppose  $(C, K)$  is the pair as above corresponding to  $L$ . Then the above claim implies  $\partial L = K$ . Let  $B$  be the  $v$ -branch corresponding to  $(C, K)$ . Then  $\partial B = \Delta(K) = \Delta(\partial L)$ . Now we prove  $\Delta(\bar{L}) \subset \bar{B}$ . Let  $l \subset \bar{L}$  be a standard geodesic. If  $d(\Delta(l), v) \geq 2$ , by Lemma 3.4, there exists standard geodesic  $l_1$  such that  $\Delta(l_1)$  and  $\Delta(l)$  are in the same  $v$ -branch and  $l_1 \cap P_v \neq \emptyset$ . The argument in Lemma 3.9 implies that there exists an edge path  $\omega \subset X(\Gamma) \setminus P_v$  connecting a vertex in  $l$  and a vertex in  $l_1$ , thus  $l_1 \subset \bar{L}$ . It follows that  $l_1 \cap K \neq \emptyset$  and  $\pi(\Delta(l_1)) \subset C$ ,

so  $\Delta(l_1) \in B$  by Corollary 3.14 (1). Hence  $\Delta(l) \in B$ . If  $d(\Delta(l), v) = 1$ , since  $\bar{L} \cap P_v = K$ , we apply Lemma 2.4 (4) with  $C_1 = \bar{L}$  and  $C_2 = P_v$  to deduce that  $l$  stays in the  $R$ -neighbourhood of  $K$  for some  $R > 0$ , thus  $\Delta(l) \in \Delta(K) = \partial B$ . Note that  $\Delta(l) \in \bar{B}$  in both cases, so  $\Delta(\bar{L}) \subset \bar{B}$ . Now we prove  $\bar{B} \subset \Delta(\bar{L})$ . Pick vertex  $w \in \bar{B}$ . If  $w \in \partial B$ , then we are done by  $\partial B = \Delta(\partial L) \subset \Delta(\bar{L})$ . Suppose  $w \in B$ . Pick an edge  $e \subset X(\Gamma)$  which connects a point in  $L$  and a point outside  $L$  and let  $l_e$  be the standard geodesic containing  $e$ . Then  $l_e \subset \bar{L}$  by the discussion in the previous paragraph. Then  $\Delta(l_e) \in \bar{L} \subset \bar{B}$ . However,  $\Delta(l_e) \notin \partial B$ , hence  $\Delta(l_e) \in B$ . The argument in Lemma 3.9 implies that there exists an edge path outside  $P_v$  connecting a vertex in  $l_e$  and a vertex in  $P_w$ . Thus  $w \in \Delta(\bar{L})$ . In summary, each vertex of  $\bar{B}$  is in  $\Delta(\bar{L})$ . Since  $\bar{L}$  is convex,  $\Delta(\bar{L})$  is a full subcomplex by Lemma 3.25, then  $\bar{B} \subset \Delta(\bar{L})$ .

To see (4), let  $\{\Delta_\lambda\}_{\lambda \in \Lambda}$  be the collection of maximal simplexes in  $\mathcal{P}(\Gamma)$  such that  $\Delta_\lambda \cap B \neq \emptyset$  and let  $\{F_\lambda\}_{\lambda \in \Lambda}$  be the collection of maximal standard flats such that  $\Delta(F_\lambda) = \Delta_\lambda$ . We claim the Hausdorff distance  $d_H(L, \cup_{\lambda \in \Lambda} F_\lambda) < \infty$ . Note that  $\Delta_\lambda \subset \bar{B}$ , hence  $F_\lambda \subset \bar{L}$  by (3) and the maximality of  $F_\lambda$ . Pick an arbitrary vertex  $x \in L$  and let  $l_x$  be a standard geodesic such that  $d(\pi(\Delta(l_x)), \bar{v}) \geq 2$  and  $x \in l_x$ . Then  $d(\Delta(l_x), v) \geq 2$ . Hence  $l_x \cap P_v$  is at most one point. It follows from the proof of (3) that  $l_x \subset \bar{L}$  and  $\Delta(l_x) \subset B$ . Thus there exists  $\lambda_0 \in \Lambda$  such that  $x \in l_x \subset F_{\lambda_0}$ . Thus  $L$  is contained in some neighbourhood of  $\cup_{\lambda \in \Lambda} F_\lambda$ . However,  $d_H(L, \bar{L}) < \infty$  by (1) and (2), hence the claim follows. Let  $B' = q_*(B)$  and  $L'$  be the component of  $X(\Gamma') \setminus P_{q_*(v)}$  corresponding to  $B'$  (note that  $\Gamma'$  is also of weak type II by Corollary 3.24). By Lemma 2.19, for each  $\lambda \in \Lambda$ , there exists a unique maximal standard flat  $F'_\lambda \subset X(\Gamma')$  such that  $d_H(q(F_\lambda), F'_\lambda) < C$  ( $C$  is independent of  $\lambda$ ). Note that we also have  $d_H(L', \cup_{\lambda \in \Lambda} F'_\lambda) < \infty$ , so  $d_H(q(L), L') < \infty$ .

Now we prove (5). It follows from Lemma 3.17, Lemma 2.19 and Lemma 3.18 that  $St(\bar{v})$  is a stable subgraph. Hence  $d_H(P_v, P_{q_*(v)}) < \infty$ . Let  $K' = P_{q_*(v)} \cap \bar{L}'$ . Then  $K'$  is a  $q_*(v)$ -peripheral subcomplex by (1) and (2), hence is a standard subcomplex. Recall that  $K = P_v \cap L$ , so  $d_H(q(K), K') < \infty$  by Lemma 2.4 (4).  $\square$

### 3.3. RAAG of weak type I.

**Definition 3.27.** A finite simplicial graph  $\Gamma$  is of *weak type I* if:

- (1)  $\Gamma$  is of weak type II.
- (2)  $\Gamma$  does not contain any separating closed star.

$G(\Gamma)$  is of *weak type I* if  $\Gamma$  is of weak type I.

It is immediate from the definition that if  $\Gamma = \Gamma_1 \circ \Gamma_2$ , then  $\Gamma$  is of weak type I if and only if  $\Gamma_1$  and  $\Gamma_2$  are of weak type I.

**Lemma 3.28.**  $G(\Gamma)$  is of weak type I if and only if

- (1)  $\Gamma$  does not contain any separating closed star.
- (2) There does not exist vertices  $\bar{v}, \bar{w} \in \Gamma$  such that  $d(\bar{v}, \bar{w}) = 2$  and  $\Gamma = St(\bar{v}) \cup St(\bar{w})$ .

Thus Definition 1.1 and Definition 3.27 are consistent.

*Proof.* For the only if direction, note that if  $\Gamma = St(\bar{v}) \cup St(\bar{w})$  with  $d(\bar{v}, \bar{w}) = 2$ , then  $lk(\bar{v}) \cap lk(\bar{w})$  separates  $\Gamma$ . For the if direction, we follow the argument in [Hua14a, Theorem 5.3]. Suppose there exist vertices  $\bar{v}_1$  and  $\bar{v}_2$  such that  $lk(\bar{v}_1) \cap lk(\bar{v}_2)$  separates  $\Gamma$ . And let  $\{C_j\}_{j=1}^d$  be the connected components of  $F(\Gamma) \setminus lk(\bar{v}_1) \cap lk(\bar{v}_2)$ .

Then at most one of  $C_j$  is contained in  $St(\bar{v}_1)$ . If  $d \geq 3$ ,  $St(\bar{v}_1)$  would separate  $F(\Gamma)$ , contradiction. Suppose  $d = 2$ . At least one of  $C_1$  and  $C_2$  is inside  $St(\bar{v}_1)$ , otherwise  $St(\bar{v}_1)$  will separate  $F(\Gamma)$ . We assume  $C_1 \subset St(\bar{v}_1)$ . Thus  $\bar{v}_2 \in C_2$ . Similarly, at least one of  $C_1$  and  $C_2$  is inside  $St(\bar{v}_2)$ . So we must have  $C_2 \subset St(\bar{v}_2)$ . Hence  $F(\Gamma) = St(\bar{v}_1) \cup St(\bar{v}_2)$ , contradiction again.  $\square$

**Theorem 3.29.** *Let  $\Gamma_1$  be of weak type I. Then any simplicial isomorphism  $s : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$  is visible. In particular, if  $q : G(\Gamma_1) \rightarrow G(\Gamma_2)$  is a quasi-isometry, then  $q$  will induce a visible map  $q_* : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$ . In this case,  $\Gamma_2$  is of weak type II, hence  $\text{Out}(\Gamma_2)$  does not contain non-adjacent transvections.*

*Proof.* Let  $p, \{F_i\}_{i=1}^n$  and  $\{F'_i\}_{i=1}^n$  be as in Definition 2.24. Suppose  $s$  is not visible. Then there exist  $i \neq j$  and a hyperplane  $h$  separating  $F'_i$  and  $F'_j$ . Let  $l'$  be a standard geodesic dual to  $h$  and let  $v' = \Delta(l')$ . Then there exists  $v'_1 \in \Delta(F'_i)$  and  $v'_2 \in \Delta(F'_j)$  such that they are in different  $v'$ -tiers. Thus  $St(v')$  separates  $v'_1$  from  $v'_2$  by Lemma 3.1. Let  $v = s^{-1}(v')$ . Then  $(F(\Gamma))_p \setminus St(v)$  is disconnected, hence  $v \in (F(\Gamma))_p$  by Remark 3.8. This would imply  $F(\Gamma)$  has a separating closed star, which is a contradiction. The second statement follows from Lemma 3.19.  $\square$

**Theorem 3.30.** *Suppose  $G(\Gamma_1)$  and  $G(\Gamma_2)$  are groups of weak type I. Then they are quasi-isometric if and only if they are isomorphic.*

*Proof.* Let  $q : G(\Gamma_1) \rightarrow G(\Gamma_2)$  be a quasi-isometry. By Theorem 3.29,  $q$  induces a visible simplicial isomorphism  $q_* : \mathcal{P}(\Gamma_1) \rightarrow \mathcal{P}(\Gamma_2)$ . Pick vertex  $x_1 \in X(\Gamma_1)$ . Then the visibility implies  $q_*((F(\Gamma_1))_{x_1}) \subset (F(\Gamma_2))_{x_2}$  for some vertex  $x_2 \in X(\Gamma_2)$ . This induces a graph embedding  $\Gamma_1 \rightarrow \Gamma_2$ . By consider the quasi-isometry inverse of  $q$ , we obtain another graph embedding  $\Gamma_2 \rightarrow \Gamma_1$ . Hence  $\Gamma_1 \cong \Gamma_2$  and  $G(\Gamma_1) \cong G(\Gamma_2)$ .  $\square$

Though the definition of weak type I looks technical, it is actually a natural condition to consider for the following reason:

**Theorem 3.31.** *The following are equivalent:*

- (1)  $G(\Gamma)$  is of weak type I.
- (2) There does not exist vertex  $x \in X(\Gamma)$  and vertex  $v \in \mathcal{P}(\Gamma)$  such that  $St(v)$  separates  $(F(\Gamma))_x$ .
- (3) Every element in  $\text{Aut}(\mathcal{P}(\Gamma))$  is visible.

*Proof.* (1)  $\Leftrightarrow$  (2) follows from Lemma 3.32 below and (2)  $\Rightarrow$  (3) follows from the proof of Theorem 3.29. It suffices to prove (3)  $\Rightarrow$  (2). We argue by contradiction and suppose  $v_1, v_2$  are vertices in different components of  $(F(\Gamma))_x \setminus St(v)$ . Then Lemma 3.9 implies  $v_1$  and  $v_2$  are in different  $v$ -branches. For  $i = 1, 2$ , let  $B_i$  be the  $v$ -branch that contains  $v_i$ . Let  $l$  be a standard geodesic such that  $\Delta(l) = v$  and  $x \in l$  and pick  $\alpha \in G(\Gamma)$  to be a non-trivial element such that  $\alpha(l) = \alpha$ . Let  $\alpha_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  be the induced map. Then  $\alpha_*$  fixes every point in  $St(v)$ . Thus there exists  $f \in \text{Aut}(\mathcal{P}(\Gamma))$  such that (1)  $f$  fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (B_1 \cup \alpha_*(B_1))$ ; (2)  $f|_{B_1} = \alpha_*|_{B_1}$  and  $f|_{\alpha_*(B_1)} = \alpha_*^{-1}|_{\alpha_*(B_1)}$ .

We claim  $f$  is not visible. To see this, for  $i = 1, 2$ , pick maximal standard flat  $F_i$  such that  $x \in F_i$  and  $v_i \in \Delta(F_i)$ . Then  $f(\Delta(F_1)) = \alpha_*(\Delta(F_1))$  and  $f(\Delta(F_2)) = \Delta(F_2)$ , thus the maximal standard flats corresponding to  $f(\Delta(F_1))$  and  $f(\Delta(F_2))$  are separated by a hyperplane dual to  $l$ , hence have empty intersection.  $\square$



**Lemma 3.32.**  $\Gamma$  is of weak type II if and only if there does not exist vertex  $x \in X(\Gamma)$  and vertex  $v \in \mathcal{P}(\Gamma) \setminus (F(\Gamma))_x$  such that  $(F(\Gamma))_x \setminus St(v)$  is disconnected.

*Proof.* By remark 3.8, it suffices to prove the if direction. Suppose  $\Gamma$  is not of weak type II. Let  $\{\bar{v}_i\}_{i=1}^3, \{v_i\}_{i=1}^3$  and  $x_0 \in X(\Gamma)$  be as in Lemma 3.22. For  $i = 1, 2$ , let  $l_i$  be the standard geodesic such that  $x_0 \in l$  and  $\Delta(l_i) = v_i$ . Pick vertex  $x'_0 \neq x_0$  in  $l_1$  and let  $l'_2$  be the standard geodesic such that  $x'_0 \in l'_2$  and  $\pi(\Delta(l'_2)) = \bar{v}_2$ . Then  $d(v'_2, v_1) = 2$  where  $v'_2 = \Delta(l'_2)$ , in particular  $x_0 \notin P_{v'_2}$ , hence  $v'_2 \notin (F(\Gamma))_{x_0}$ .

Since  $P_{v'_2}$  and  $x_0$  are separated by some hyperplane dual to  $l_1$ , thus by Lemma 2.12,  $St(v'_2) \cap (F(\Gamma))_{x_0} \subset St(v_1)$ . Recall that  $d(\bar{v}_3, \bar{v}_1) \geq 2$ , then  $v_3 \in (F(\Gamma))_{x_0} \setminus St(v_1)$ . It follows that  $v_3 \notin St(v'_2)$ .

We claim that  $v_3$  and  $v_1$  are in different components of  $\mathcal{P}(\Gamma) \setminus St(v'_2)$ , which then implies  $(F(\Gamma))_{x_0} \setminus St(v'_2)$  is disconnected. Lemma 3.22 already implies  $v_1$  and  $v_3$  are separated by  $lk(v_1) \cap lk(v_2)$ . Let  $\alpha \in G(\Gamma)$  be the left translation such that  $\alpha(x_0) = x'_0$ . Then  $\alpha(l_2) = l'_2$ . Now we pass to the induced action  $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$ , then  $\alpha(v_2) = v'_2$ . Since  $\alpha$  fixes  $St(v_1)$ , we have  $lk(v_1) \cap lk(v_2) = \alpha(lk(v_1) \cap lk(v_2)) = lk(v_1) \cap lk(v'_2)$ . So  $lk(v_1) \cap lk(v'_2)$  separates  $v_1$  from  $v_3$  and the claim follows.  $\square$

#### 4. RIGIDITY AND FLEXIBILITY OF RAAG OF WEAK TYPE I

**4.1. An atlas for RAAG.** Let  $G(\Gamma)$  be a RAAG of weak type I with trivial centre and suppose  $G(\Gamma) \neq \mathbb{Z}$ . We identify  $G(\Gamma)$  with the 1-skeleton of  $X(\Gamma)$  and define a *standard flat* in  $G(\Gamma)$  to be the vertex set of some standard flat in  $X(\Gamma)$ .

Theorem 3.29 implies there is a homomorphism  $s : Aut(P(\Gamma)) \rightarrow Perm(G(\Gamma))$ , where  $Perm(G(\Gamma))$  is the permutation group of elements in  $G(\Gamma)$ . Note that images of  $s$  preserve maximal standard flats. However, this may not be true for all standard flats, since adjacent transvections are allowed in  $Out(G(\Gamma))$ .

Let  $\mathcal{P}(\Gamma)$  be the extension complex and let  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  be label-preserving simplicial map defined in Section 2.3. Note that for any vertex  $x \in X(\Gamma)$ ,  $\pi$  induces an isomorphism  $(F(\Gamma))_x \rightarrow F(\Gamma)$ . This motivates the following definition.

**Definition 4.1** (Coherent labelling). A *coherent labelling* of  $G(\Gamma)$  is a simplicial map  $L : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  such that  $L$  induces an isomorphism  $(F(\Gamma))_x \rightarrow F(\Gamma)$  for every vertex  $x \in X(\Gamma)$ .

Assume  $n = \dim(X(\Gamma))$ . Let  $\mathcal{F}(\Gamma)$  be the collection of stable standard flat in  $X(\Gamma)$  and let  $\mathcal{F}_k(\Gamma)$  be the collection of  $k$ -flats in  $\mathcal{F}(\Gamma)$ . Define  $\mathcal{F}_{<k}(\Gamma) := \cup_{i=1}^{k-1} \mathcal{F}_i(\Gamma)$ . Here we are considering the set itself, not the coarse equivalent classes of the sets (compared to Theorem 2.20). Recall that we use  $v(K)$  to denote the set of vertices in a subset  $K$  of some polyhedral complex. An  $L$ -atlas is a coherent labelling  $L : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  together with a collection of bijections

$$\{v(F) \rightarrow \mathbb{Z}^{v(L(\Delta(F)))}\}_{F \in \mathcal{F}_k(\Gamma), 1 \leq k \leq n}$$

with the following compatibility condition: pick  $F_1 \in \mathcal{F}_m(\Gamma)$  and  $F_2 \in \mathcal{F}_l(\Gamma)$  with  $F_1 \subset F_2$ , let  $f : v(F_2) \rightarrow \mathbb{Z}^{v(L(\Delta(F_2)))}$  and  $g : v(F_1) \rightarrow \mathbb{Z}^{v(L(\Delta(F_1)))}$  be the associated bijections. Suppose  $p : \mathbb{Z}^{v(L(\Delta(F_2)))} \rightarrow \mathbb{Z}^{v(L(\Delta(F_1)))}$  is the natural projection. Then

- (1)  $f(v(F_1))$  is a coset of  $\mathbb{Z}^{v(L(\Delta(F_1)))}$  in  $\mathbb{Z}^{v(L(\Delta(F_2)))}$ .
- (2) The following diagram commutes up to translation:

$$\begin{array}{ccc}
v(F_1) & \xrightarrow{g} & \mathbb{Z}^{v(L(\Delta(F_1)))} \\
\downarrow i & & \uparrow p \\
v(F_2) & \xrightarrow{f} & \mathbb{Z}^{v(L(\Delta(F_2)))}
\end{array}$$

Here  $i$  is the inclusion map.

$L$ -atlas  $\mathcal{A}_L$  and  $L'$ -atlas  $\mathcal{A}_{L'}$  are *equal up to translations* if and only if  $L = L'$  and the bijections in  $\mathcal{A}_L$  and  $\mathcal{A}_{L'}$  agree up to translation. We will write  $\mathcal{A}_L \stackrel{e}{=} \mathcal{A}_{L'}$  in this case. Pick  $\alpha \in \text{Aut}(\mathcal{P}(\Gamma))$  and let  $\alpha_* : G(\Gamma) \rightarrow G(\Gamma)$  be the bijection induced by  $\alpha$ . Recall that  $\alpha_*$  preserves stable standard flats. The *pull-back* of an  $L$ -atlas  $\mathcal{A}_L$  under  $\alpha$ , denoted by  $\alpha^*(\mathcal{A}_L)$ , is defined to be the  $(\alpha \circ L)$ -atlas with its charts being the pull-backs of charts of  $\mathcal{A}_L$  under  $\alpha_*$ .

Recall that we label each circle in  $S(\Gamma)$  be a generator of  $G(\Gamma)$  and fix an orientation for each circle. This lifts to  $G(\Gamma)$ -invariant labelling and orientation of edges in  $X(\Gamma)$ . Moreover, we have induced action  $G(\Gamma) \curvearrowright \mathcal{P}(\Gamma)$  and induced  $G(\Gamma)$ -invariant labelling of vertices in  $\mathcal{P}(\Gamma)$ . This leads to a naturally defined  $L$ -atlas as follows.

Let  $L$  be the label preserving map  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$ . For each vertex  $u \in \mathcal{P}(\Gamma)$ , we pick a standard geodesic  $l \subset X(\Gamma)$  such that  $\Delta(l) = u$ , and identify vertices of  $l$  with  $\mathbb{Z}^u$  in an orientation-preserving way. Let  $p_u : G(\Gamma) \rightarrow \mathbb{Z}^u$  be the map induced by the  $CAT(0)$  projection from  $G(\Gamma)$  to  $l$  (recall that we have identified  $G(\Gamma)$  with vertices of  $X(\Gamma)$ , and Lemma 2.4 implies that the image of each vertex of  $X(\Gamma)$  under the  $CAT(0)$  projection is a vertex in  $l$ ). For each standard flat  $F \subset X(\Gamma)$ ,  $p_u(v(F))$  is surjective if  $u \in \Delta(F)$ , otherwise  $p_u(v(F))$  is a point. This induces a bijection  $\prod_{u \in \Delta(F)} p_u : v(F) \rightarrow \mathbb{Z}^{v(\Delta(F))}$  and we define the chart for  $F$  to be  $\prod_{u \in \Delta(F)} p_u$  post-composed with  $\mathbb{Z}^{v(\Delta(F))} \rightarrow \mathbb{Z}^{v(L(\Delta(F)))}$ . One readily verifies this atlas  $\mathcal{A}_L$  satisfies the above definition of  $L$ -atlas, moreover, the diagram in (2) commutes exactly, not up to translations. The following properties are immediate.

- (1)  $\mathcal{A}_L$  is  $G(\Gamma)$ -invariant up to translations in the sense that  $g^*(\mathcal{A}_L) \stackrel{e}{=} \mathcal{A}_L$  for all  $g \in G(\Gamma)$ . Conversely, if  $\alpha \in \text{Aut}(\mathcal{P}(\Gamma))$  satisfies  $\alpha^*(\mathcal{A}_L) \stackrel{e}{=} \mathcal{A}_L$ , then the induces map  $\alpha_* : G(\Gamma) \rightarrow G(\Gamma)$  is a left translation.
- (2)  $\mathcal{A}_L$  is unique up to translations. Since the only ambiguity in the definition of  $\mathcal{A}_L$  is the orientation-preserving identification of  $v(l)$  with  $\mathbb{Z}^u$ , which is unique up to translations.

$\mathcal{A}_L$  is called the *reference atlas*.

**Lemma 4.2.** *Let  $G(\Gamma)$  be of weak type I and pick  $F \in \mathcal{F}(\Gamma)$ . Then there exist standard flats  $\{F_i\}_{i=1}^k$  in  $F$  such that  $F$  is the convex hull of these flats and each  $F_i$  is the intersection of maximal standard flats.*

*Proof.* Pick vertex  $w \in \Gamma$ . Let  $\Gamma_w$  be the minimal stable subgraph containing  $w$  and Let  $\Gamma'_w$  be the intersection of maximal cliques that contains  $w$ . It suffices to show  $\Gamma_w = \Gamma'_w$ . Since each maximal clique is stable (Lemma 3.17 and Lemma 2.19),  $\Gamma_w \subset \Gamma'_w$ . Pick vertex  $v \in \Gamma'_w$ , then  $w^\perp \subset St(v)$ , thus  $v \in \Gamma_w$  by [Hua14a, Lemma 3.32]. It follows that  $\Gamma'_w \subset \Gamma_w$ .  $\square$

Let  $q : G(\Gamma) \rightarrow G(\Gamma')$  be a quasi-isometry and let  $s : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$  be an induced simplicial isomorphism as in Lemma 3.19. By Theorem 3.29,  $s$  induces a map  $\phi : G(\Gamma) \rightarrow G(\Gamma')$ . Note that  $\phi$  is a quasi-isometry. For every  $g' \in G(\Gamma')$ , there is left translation  $\bar{\phi}_{g'} : G(\Gamma') \rightarrow G(\Gamma')$ , which gives rise to a simplicial isomorphism

$\bar{s}_{g'} : \mathcal{P}(\Gamma') \rightarrow \mathcal{P}(\Gamma')$ . Let  $s_{g'} = s^{-1} \circ \bar{s}_{g'} \circ s$ . Then  $s_{g'}$  induces a unique bijection  $\phi_{g'} : G(\Gamma) \rightarrow G(\Gamma')$  by Theorem 3.29, moreover,

$$(4.3) \quad \bar{\phi}_{g'} \circ \phi = \phi \circ \phi_{g'}$$

In summary, we have  $G(\Gamma')$  acts on  $G(\Gamma')$ ,  $\mathcal{P}(\Gamma')$ ,  $G(\Gamma)$  and  $\mathcal{P}(\Gamma)$ .

**Lemma 4.4.**

- (1)  $\phi$  is surjective. For any  $y, y' \in G(\Gamma')$ ,  $|\phi^{-1}(y)| = |\phi^{-1}(y')| < \infty$ .
- (2) For any  $k$  and  $F \in \mathcal{F}_k(\Gamma)$ , there exists unique  $F' \in \mathcal{F}_k(\Gamma')$  such that  $\phi(v(F)) = v(F')$ . Moreover, let  $\text{Stab}(v(F'))$  and  $\text{Stab}(v(F))$  be the stabilizer of  $v(F')$  and  $v(F)$  with respect to the action  $G(\Gamma') \curvearrowright G(\Gamma')$  and  $G(\Gamma') \curvearrowright G(\Gamma)$  respectively. Then  $\text{Stab}(v(F')) = \text{Stab}(v(F))$ . In this case we will write  $F' = \phi(F)$  for simplicity.
- (3) Let  $F_1, F_2 \in \mathcal{F}_k(\Gamma)$  be parallel standard flats. Then for vertices  $x_1 \in F_1$  and  $x_2 \in F_2$ ,  $|\phi^{-1}(\phi(x_1)) \cap F_1| = |\phi^{-1}(\phi(x_2)) \cap F_2|$ .

*Proof.* Pick a reference point  $q \in \text{Im } \phi$  and let  $K_q = (F(\Gamma'))_q$ . Denote the points in  $\phi^{-1}(q)$  by  $\{p_\lambda\}_{\lambda \in \Lambda}$  and let  $K_{p_\lambda} = (F(\Gamma))_{p_\lambda}$ . Since  $\{\phi(K_{p_\lambda})\}_{\lambda \in \Lambda}$  are distinct subcomplexes of  $K_q$ ,  $\Lambda$  is a finite set. The other parts of (1) follows from (4.3).

Now we prove (2). It is clear if  $F$  is a maximal standard flat. Next we look at the case when  $F = \cap_{i=1}^h F_i$  where each  $F_i$  is a maximal standard flat. Let  $F'_i$  be maximal standard flat in  $X(\Gamma')$  such that  $\Delta(F'_i) = s(\Delta(F_i))$  for  $1 \leq i \leq h$  and let  $F' = \cap_{i=1}^h F'_i$ . Then  $\phi(v(F)) \subset v(F')$ . The stabilizer  $\text{Stab}(v(F'))$  fixes  $\Delta(F'_i)$  for all  $i$ , hence it fixes  $\Delta_i$  for all  $i$  and  $\text{Stab}(v(F')) \subset \text{Stab}(v(F))$ . Since  $\text{Stab}(v(F'))$  acts on  $v(F')$  transitively, (4.3) implies  $\phi(v(F)) = v(F')$  and  $\text{Stab}(v(F)) \subset \text{Stab}(v(F'))$ . Thus  $\text{Stab}(v(F')) = \text{Stab}(v(F))$ .

In general, by Lemma 4.2, we can assume  $F$  is the convex hull of  $F_1, F_2 \in \mathcal{F}(\Gamma)$  such that (2) is true for flats in  $F$  which are parallel to  $F_1$  or  $F_2$ . Let  $F'_i = \phi(F_i)$  for  $i = 1, 2$ . Then  $F'_1 \cap F'_2 \neq \emptyset$  and the convex hull of  $F'_1$  and  $F'_2$  is a flat  $F'$  (since  $F$  is contained in a maximal standard flat, whose image under  $\phi$  is a maximal standard flat containing  $F'_1$  and  $F'_2$ ). It follows from Lemma 2.18 that  $F' \in \mathcal{F}(\Gamma')$ . Note that any standard flat that is parallel to  $F_1$  and intersects  $F_2$  is mapped by  $\phi$  to a standard flat that is parallel to  $F'_1$  and intersects  $F'_2$ , thus  $\phi(v(F)) \subset v(F')$ . Let  $F_3 \subset F$  be a standard flat parallel to  $F_1$  and let  $F'_3 = \phi(F_3)$ . Since parallel standard flats in  $X(\Gamma')$  have the same stabilizer, we have  $\text{Stab}(v(F_1)) = \text{Stab}(v(F'_1)) = \text{Stab}(v(F'_3)) = \text{Stab}(v(F_3))$ . By considering all such  $F_3$ 's in  $F$ , we have  $\text{Stab}(v(F_1)) \subset \text{Stab}(v(F))$ . Similarly,  $\text{Stab}(v(F_2)) \subset \text{Stab}(v(F))$ , thus

$$\begin{aligned} \text{Stab}(v(F')) &= \text{Stab}(v(F'_1)) \oplus \text{Stab}(v(F'_2)) \\ &= \text{Stab}(v(F_1)) \oplus \text{Stab}(v(F_2)) \subset \text{Stab}(v(F)) \end{aligned}$$

Now we can conclude  $\phi(v(F)) = v(F')$  as before. It also follows that  $\text{Stab}(v(F)) \subset \text{Stab}(v(F'))$ , thus

$$(4.5) \quad \text{Stab}(v(F')) = \text{Stab}(v(F)).$$

Now we prove (3). Note that for a pair of parallel standard flats  $F'_1$  and  $F'_2$  in  $X(\Gamma')$ , there exists  $g' \in G(\Gamma')$  such that  $g'(v(F'_1)) = v(F'_2)$ , so by (4.3), it suffices to prove (3) in the case where  $\phi(v(F_1)) = \phi(v(F_2)) = v(F')$ . Let  $p : v(F_1) \rightarrow v(F_2)$  be the parallelism map. Denote  $p_1 = \phi|_{v(F_1)} : v(F_1) \rightarrow v(F')$  and  $p_2 = \phi|_{v(F_2)} \circ p :$

$v(F_1) \rightarrow v(F')$ . Then there exist  $L$  and  $A$  such that  $p_1$  and  $p_2$  are  $(L, A)$ -quasi-isometries and

$$(4.6) \quad d(p_1(x), p_2(x)) < A$$

for any  $x \in v(F_1)$ . Pick  $y \in v(F')$  and let  $r_i$  be the number of points  $|p_i^{-1}(y)|$  in  $p_i^{-1}(y)$  for  $i = 1, 2$  ( $r_i$  does not depend on  $y$  by previous discussion). We argue by contradiction and assume  $r_1 < r_2$ . Pick base point  $x_0 \in v(F_1)$ , let  $m = \dim(F_1)$ ,  $B_R = B(x_0, R)$  and  $K_{i,R} = p_i(B_R)$  for  $i = 1, 2$ . Then it follows from (4.6) that

$$\begin{aligned} |K_{1,R}| &\leq |N_A(K_{2,R})| = |K_{2,R}| + |N_A(K_{2,R}) \setminus K_{2,R}| \\ &\leq |K_{2,R}| + |p_2^{-1}(N_A(K_{2,R}) \setminus K_{2,R})| \leq |K_{2,R}| + |B_{LA+A+R} \setminus B_R| \\ &\leq |K_{2,R}| + CR^{m-1}(LA + A), \end{aligned}$$

here  $C$  is some constant independent of  $R$ . Thus by symmetry we have

$$(4.7) \quad ||K_{1,R}| - |K_{2,R}|| \leq CR^{m-1}(LA + A).$$

On the other hand,  $B_R \subset p_i^{-1}(K_{i,R}) \subset B_{R+A}$  for  $i = 1, 2$ , thus

$$(4.8) \quad CR^m \leq |p_i^{-1}(K_{i,R})| = r_i |K_{i,R}| \leq C(R + A)^m$$

for  $i = 1, 2$ . Now (4.7) and (4.8) imply

$$\begin{aligned} CR^m/r_1 - C(R + A)^m/r_2 &\leq |K_{1,R}| - |K_{2,R}| \\ &\leq ||K_{1,R}| - |K_{2,R}|| \leq CR^{m-1}(LA + A). \end{aligned}$$

Since  $r_1 < r_2$ , we will get a contradiction when  $R \rightarrow \infty$ .  $\square$

**Lemma 4.9.** *Given  $L$ -atlas  $\mathcal{A}_L$  and  $L'$ -atlas  $\mathcal{A}_{L'}$ , there exists  $\alpha \in \text{Aut}(\mathcal{P}(\Gamma))$  such that  $\alpha^*(\mathcal{A}_{L'}) \stackrel{e}{=} \mathcal{A}_L$ . Moreover, there is a 1-1 correspondence between elements in  $\text{Aut}(\mathcal{P}(\Gamma))$  and the following set of information:*

- (1) *A base point  $p \in G(\Gamma)$ .*
- (2) *A class of  $L$ -atlases which are equal up to translations.*

We follow [Hua14a, Lemma 5.7].

*Proof.* We prove the first part of the lemma. Pick  $v \in G(\Gamma)$ , set  $\alpha'(e) = v$ . For  $u \in G(\Gamma)$ , pick a word  $w_u = a_1 a_2 \cdots a_n$  representing  $u$ , let  $u_i = a_1 a_2 \cdots a_i$  for  $1 \leq i \leq n$  and  $u_0 = e$ . We define  $q_i = \alpha'(a_1 a_2 \cdots a_i) \in G(\Gamma)$  inductively as follows: set  $q_0 = v$  and suppose  $q_{i-1}$  is already defined. Let  $F_i \in \mathcal{F}(\Gamma)$  be a standard flat containing  $u_{i-1}$  and  $u_i$  and let  $F'_i$  be the unique standard flat such that  $q_{i-1} \in F'_i$  and  $L'(\Delta(F'_i)) = L(\Delta(F_i))$ . There is a natural identification  $f_i : F_i \rightarrow F'_i$  via the charts  $F_i \rightarrow \mathbb{Z}^{v(L(\Delta(F_i)))}$  and  $F'_i \rightarrow \mathbb{Z}^{v(L'(\Delta(F'_i)))} = \mathbb{Z}^{v(L(\Delta(F_i)))}$ . We can assume  $f_i(u_{i-1}) = q_{i-1}$  by post-composing a translation, then define  $q_i = f_i(u_i)$ .  $q_i$  does not depend on the choice of  $F_i$  by the compatibility condition (2).

We claim for any other word  $w'_u$  representing  $u$ ,  $\phi(w_u) = \phi(w'_u)$ , hence there is a well-defined map  $\phi : G(\Gamma) \rightarrow G(\Gamma)$ . To see this, recall that one can obtain  $w_u$  from  $w'_u$  by performing the following two basic moves:

- (1)  $w_1 a a^{-1} w_2 \rightarrow w_1 w_2$ .
- (2)  $w_1 a b w_2 \rightarrow w_1 b a w_2$  when  $a$  and  $b$  commutes.

It is clear that  $\phi(w_1aa^{-1}w_2) = \phi(w_1w_2)$  and it suffices to show  $\alpha'(ab) = \alpha'(ba)$  where  $a$  and  $b$  are mutually commuting generators. Let  $F$  be a maximal standard flat that contains  $e, a$  and  $b$ , we could always choose  $F$  in the definition of  $\alpha'(ab)$  or  $\alpha'(ba)$ , thus they are equal.

By switching the role of  $\mathcal{A}_L$  and  $\mathcal{A}_{L'}$ , we can define another  $\alpha'' : G(\Gamma) \rightarrow G(\Gamma)$  which maps  $v$  to  $e$ . It is not hard to check  $\alpha'$  and  $\alpha''$  are mutual inverse. Thus  $\alpha'$  is bijective, moreover,  $\alpha'$  preserves  $\mathcal{F}(\Gamma)$ . To define  $\alpha$ , pick vertex  $v \in \mathcal{P}(\Gamma)$ , let  $\Delta$  be a maximal simplex containing  $v$ . Take  $F \subset X(\Gamma)$  to be the flat such that  $\Delta(F) = \Delta$  and take  $F'$  to be the maximal standard flat such that  $\alpha(v(F)) = v(F')$ , we set  $\alpha(v)$  to be the unique point such that  $\alpha(v) \in \Delta(F')$  and  $L(v) = L'(\alpha(v))$ . It is clear that  $L = L' \circ \alpha$ ,  $\alpha'$  is induced by  $\alpha$  and  $\alpha'$  pulls back the charts up to translations, so  $\alpha^*(\mathcal{A}_{L'}) \stackrel{e}{=} \mathcal{A}_L$ .

Now we prove the second part of the lemma. Let  $\mathcal{A}_L$  be the reference atlas. By Theorem 3.29, any element  $\beta \in \text{Aut}(\mathcal{P}(\Gamma))$  induces a bijection  $\beta_* : G(\Gamma) \rightarrow G(\Gamma)$  which preserves  $\mathcal{F}(\Gamma)$ . Thus the pull-back  $\beta^*(\mathcal{A}_L)$  is a  $(\beta \circ L)$ -atlas. So for each  $\beta$ , we associated a pair  $(\beta_*(e), \beta^*(\mathcal{A}_L))$  where  $e \in G(\Gamma)$  is the identity element. On the other hand, given a point  $p \in G(\Gamma)$  and a class of  $L'$ -atlases equal up to translation, the previous discussion produces an automorphism  $\beta$  of  $\mathcal{P}(\Gamma)$  such that  $\beta^*(\mathcal{A}_L)$  belongs to this class and  $\beta_*(e) = p$ . One readily verifies this is a 1-1 correspondence.  $\square$

One may notice that most of the above proof does not use anything about the stability of flats. Indeed, we can formulate a version for general right-angled Artin group. We call a collection of charts a *reduced atlas* if they are only defined for standard flats which are intersections of maximal standard flats. The compatibility of these charts and the coherence of the labelling is required as before. Let  $\text{Aut}_v(\mathcal{P}(\Gamma))$  be the subgroup of  $\text{Aut}(\mathcal{P}(\Gamma))$  made of visible elements. Note that distinct visible elements may give rise to the same element in  $\text{Perm}(G(\Gamma))$ .

**Lemma 4.10.** *If  $G(\Gamma)$  is a right-angled Artin group with trivial centre (not necessarily of weak type I), then there is a 1-1 correspondence between elements in  $\text{Aut}_v(\mathcal{P}(\Gamma))$  and the following set of information:*

- (1) A base point  $p \in G(\Gamma)$ .
- (2) A class of reduced atlases which are equal up to translations.

We can prove the lemma as before. Let  $F_1$  and  $F_2$  be flats in a reduce atlas and assume the convex hull of them is another flat  $F$ . Then  $F$  has a natural chart and we have a compatible set of charts for a larger collection of flats. Lemma 4.2 implies that in the case of group of weak type I, this larger collection is exactly the collection of stable flats, so Lemma 4.9 and Lemma 4.10 are consistent.

**4.2. Shear standard flats.** We return to the case where  $G(\Gamma)$  is a RAAG of weak type I with trivial centre. Let  $\mathcal{A}_{L'}$  be the reference atlas for  $G(\Gamma')$ . Let  $q, s, s_{g'}, \bar{s}_{g'}, \phi, \phi_{g'}$  and  $\bar{\phi}_{g'}$  be as in the discussion before Lemma 4.4. We will also be using actions of  $G(\Gamma')$  on  $\mathcal{P}(\Gamma'), G(\Gamma'), \mathcal{P}(\Gamma)$  and  $G(\Gamma)$  discussed over there.

**Lemma 4.11.** *There exists a coherence labelling  $L : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  for  $G(\Gamma)$  which is invariant under the action  $G(\Gamma') \curvearrowright \mathcal{P}(\Gamma)$ .*

This lemma is a consequence of Lemma 3.29. A detailed proof is given in [Hua14a, Lemma 5.9].

Our goal is to construct an  $L$ -atlas  $\bar{\mathcal{A}}_L$  for  $G(\Gamma)$  such that

- (1)  $\bar{\mathcal{A}}_L$  is  $G(\Gamma')$ -invariant up to translations. In the light of Lemma 4.11, it suffices to show charts of  $\bar{\mathcal{A}}_L$  are  $G(\Gamma')$ -invariant up to translations.
- (2) Charts in  $\bar{\mathcal{A}}_L$  are “pull-backs” of charts in  $\mathcal{A}_{L'}$  under  $\phi : G(\Gamma) \rightarrow G(\Gamma')$ .

By induction, we assume the charts are already defined for standard flats in  $\mathcal{F}(\Gamma)$  of dimension  $\leq k-1$  such that

- (1) The charts are compatible and  $G(\Gamma')$ -invariant up to translations.
- (2) (inverse images are boxes) Given  $F \in \mathcal{F}_{<k}(\Gamma)$ , let  $F' = \phi(F)$  and let  $s_0 : v(L(\Delta(F))) \rightarrow v(L'(\Delta(F')))$  be the bijection induced by  $s$ . Suppose  $\bar{h}$  and  $h'$  are charts for  $F$  and  $F'$  respectively. Then  $\varphi = h' \circ \phi \circ \bar{h}$  admits splitting  $\varphi = \prod_{w \in v(L(\Delta(F)))} \varphi_w$  where  $\varphi_w : \mathbb{Z}^w \rightarrow \mathbb{Z}^{s_0(w)}$  is of form  $\varphi_w(a) = \lfloor a/d_w \rfloor + r_w$  for some integers  $r_w$  and  $d_w$  ( $d_w > 0$ ).
- (3) (extension condition) For  $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$  such that  $\phi(v(F_1)) = \phi(v(F_2)) = v(F')$ , there is a bijection  $f : v(F_1) \rightarrow v(F_2)$  such that
  - (a)  $\phi(x) = \phi \circ f(x)$  for any  $x \in v(F_1)$ .
  - (b)  $\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$  is a translation (here  $\bar{h}_i : v(F_i) \rightarrow \mathbb{Z}^{v(L(\Delta(F_i)))}$  are charts).
  - (c) Let  $F \in \mathcal{F}(\Gamma)$  such that  $F_1 \cap F \neq \emptyset$ ,  $F_2 \cap F \neq \emptyset$  and the convex hull of  $F_1$  and  $F$  is a flat. Then  $f(v(F_1 \cap F)) = v(F_2 \cap F)$ .

**Remark 4.12.** Note that only requiring condition (1) is not enough since the compatibility of existing charts does not imply that we can add more charts in a compatible way to obtain an atlas, thus we need condition (3). Condition (2) says that the charts in  $\bar{\mathcal{A}}_L$  are nice pull-backs of the charts in  $\mathcal{A}_{L'}$  (compare with Equation (5.13) of [Hua14a]).

For  $i = 1, 2$ , let  $\varphi_i = h' \circ \phi \circ \bar{h}_i^{-1}$ . Then (2) and (3) imply that  $\varphi_1^{-1}(y)$  and  $\varphi_2^{-1}(y)$  are boxes of the same size for any  $y \in \mathbb{Z}^{v(L(\Delta(F')))}$ . Thus (3a) and (3b) uniquely determine the map  $f$  and we call  $f$  a *charts-induced identification* (CII) between  $v(F_1)$  and  $v(F_2)$ .

**Lemma 4.13.** *The map  $f$  is  $\text{Stab}(v(F'))$ -equivariant.*

*Proof.* Recall that  $\text{Stab}(v(F')) = \text{Stab}(v(F_1)) = \text{Stab}(v(F_2))$  by Lemma 4.4. By (1), the induced action of  $\text{Stab}(v(F'))$  on the range of  $\bar{h}_1$  (or  $\bar{h}_2$ ) is an action by translations, moreover, this action is completely determined by the size of the box  $\varphi_1^{-1}(y)$  (or  $\varphi_2^{-1}(y)$ ). It follows from (3a) and (3b) that  $\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$  is  $\text{Stab}(v(F'))$ -equivariant. Then the lemma follows.  $\square$

**Remark 4.14.** Recall that in [Hua14a, Lemma 5.7], all CIIs between standard geodesics are induced by parallelism. This is relaxed to (3c), which says CII is induced by parallelism whenever it has to be.

Let  $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$  be two parallel elements. If  $F'_1 = \phi(F_1)$  and  $F'_2 = \phi(F_2)$ , then  $F'_1$  and  $F'_2$  are parallel. Let  $p : v(F'_1) \rightarrow v(F'_2)$  be the map induced by parallelism, and let  $g'$  be the unique element in  $G(\Gamma')$  such that  $\phi_{g'}|_{v(F'_1)} = p$ . Suppose  $F_{21} = \phi_{g'}(F_1)$ . Then  $\phi(F_{21}) = \phi(F_2)$  by (4.3). We define the CII  $f : v(F_1) \rightarrow v(F_2)$  by  $f = f_1 \circ \phi_{g'}$  where  $f_1$  is the CII between  $v(F_{21})$  and  $v(F_2)$ .

**Lemma 4.15.** *The CII  $f$  is  $\text{Stab}(v(F_1))$ -equivariant.*

*Proof.* Note that  $\text{Stab}(v(F_1)) = \text{Stab}(v(F_2)) = \text{Stab}(v(F'_1)) = \text{Stab}(v(F'_2))$ . Since  $g'$  commutes with any element in  $\text{Stab}(v(F_1))$ ,  $\phi_{g'}$  is  $\text{Stab}(v(F_1))$ -equivariant. By Lemma 4.13,  $f_1$  is  $\text{Stab}(v(F_1))$ -equivariant. Thus the lemma follows.  $\square$

We claim  $f$  satisfies all the properties in (3) with (3a) replaced by  $\bar{\phi}_{g'} \circ \phi = \phi \circ f$ . By condition (1), it suffices to check (3c). Let  $F' = \phi(F)$ . Then the convex hull of  $F'$  and  $F'_1$  (or  $F'_2$ ) is also a flat, thus  $\bar{\phi}_{g'}(v(F')) = v(F')$  and (4.5) implies  $\phi_{g'}(v(F)) = v(F)$ . It follows that  $\phi_{g'}(v(F \cap F_1)) = v(F \cap F_{21})$ . Note that  $F_{21}$  and  $F_2$  are in the convex hull of  $F_1$  and  $F$ , thus  $f_1(v(F \cap F_{21})) = v(F \cap F_2)$  by (3c), which implies  $f(v(F \cap F_1)) = v(F \cap F_2)$ . The claim and induction assumption (2) also imply the following result:

**Lemma 4.16.**  *$f$  is uniquely characterized by the following two properties:*

- $\bar{h}_2 \circ f \circ \bar{h}_1^{-1}$  is a translation.
- $\phi \circ f(x) = p \circ \phi(x)$  for any  $x \in v(F_1)$ , where  $p : v(F'_1) \rightarrow v(F'_2)$  is the parallelism map.

**Lemma 4.17.** *Let  $\{F_i\}_{i=1}^4 \subset \mathcal{F}_{<k}(\Gamma)$  such that  $F_1, F_2$  and  $F_3$  are parallel. Suppose  $f_{ij}$  is the CII between  $v(F_i)$  and  $v(F_j)$  and  $h_i$  is the chart for  $F_i$ . Then*

- (1)  $f_{13} = f_{23} \circ f_{12}$  (it is a consequence of Lemma 4.16).
- (2) If  $F_4 \subset F_1$ , then  $f_{12}(v(F_4))$  is the vertex set of some standard flat and  $f_{12}|_{F_4}$  is the CII between  $v(F_4)$  and  $f_{12}(v(F_4))$ .
- (3) If  $F_i \subset F_4$  for  $i = 1, 2$ , then  $f_{12}$  coincides with the map induced by parallelism between  $\bar{h}_4(v(F_1))$  and  $\bar{h}_4(v(F_2))$  in  $\mathbb{Z}^{v(L(\Delta(F_4)))}$ .
- (4) CIIs are  $G(\Gamma')$ -invariant. Namely for any  $g' \in G(\Gamma')$ , the CII between  $\phi_{g'}(F_1)$  and  $\phi_{g'}(F_2)$  is given by  $\phi_{g'} \circ f_{12} \circ \phi_{g'}^{-1}$ .
- (5) (3c) is true with  $f$  replaced by  $f_{12}$  (we do not need  $\phi(v(F_1)) = \phi(v(F_2))$ ).

*Proof.* To see (2), by the compatibility of charts, there is a 1-1 correspondence between standard flats in  $F_2$  that are parallel to  $F_4$  and cosets of  $\mathbb{Z}^{v(L(\Delta(F_4)))}$  in  $\mathbb{Z}^{v(L(\Delta(F_2)))}$ , but  $\bar{h}_2(f_{12}(v(F_4)))$  is such coset by Lemma 4.16, thus (2) is true. (3) follows from induction assumption (2) and Lemma 4.16. (4) follows from induction assumption (1) and Lemma 4.16. (5) is the claim before Lemma 4.16.  $\square$

Pick  $F_1, F \in \mathcal{F}(\Gamma)$  with  $\dim(F_1) < k$  and  $F_1 \subset F$ , we define a map  $\pi_1 : v(F) \rightarrow v(F_1)$  as follows. For standard flat  $K \subset F$  with  $K$  parallel to  $F_1$ , we set  $\pi|_{v(K)} = f_K$  where  $f_K : v(K) \rightarrow v(F_1)$  is the CII. We call  $\pi_1$  a *charts-induced projection* (CIP).

**Lemma 4.18.** *Let  $F'_1 = \phi(F_1)$  and  $F' = \phi(F)$ . Suppose  $F'_2$  is an orthogonal complement of  $F'_1$  in  $F'$  and  $h_1$  is the chart for  $F_1$ . Then*

- (1)  $\pi_1 \circ \bar{h}_1$  is  $\text{Stab}(v(F'_2))$ -invariant and  $\text{Stab}(v(F'_1))$ -invariant up to translation, hence is  $\text{Stab}(v(F'))$ -invariant up to translation.
- (2) Pick  $F_3 \in \mathcal{F}(\Gamma)$  such that  $F_3 \subset F_1$ . Let  $\pi_3 : v(F) \rightarrow v(F_3)$  and  $\pi_{13} : v(F_1) \rightarrow v(F_3)$  be CIPs. Then  $\pi_3 = \pi_{13} \circ \pi_1$ .
- (3) Assume  $\dim(F) < k$  and let  $h$  be the chart for  $F$ . Then  $\pi_1$  coincides with the map induced by the natural projection from  $h(F)$  to  $h(F_1)$  in  $\mathbb{Z}^{v(L(\Delta(F)))}$ .
- (4) Suppose  $\pi'_1$  is the orthogonal projection  $v(F') \rightarrow v(F'_1)$ . Then  $\phi \circ \pi_1(x) = \pi'_1 \circ \phi(x)$  for any  $x \in v(F)$ .
- (5) Let  $F_3 \in \mathcal{F}(\Gamma)$  be a standard flat in  $F$ . Then there exists stable standard flat  $F_4 \in F_1$  such that  $\pi_1(v(F_3)) = v(F_4)$  ( $F_4$  could be a point). Moreover, let  $\pi_4 : F \rightarrow F_4$  be the CIP. Then  $\pi_1|_{v(F_3)} = \pi_4|_{v(F_3)}$ .

*Proof.* To see (1), note that any element in  $\text{Stab}(v(F'_2))$  maps  $F'_1$  to a flat parallel to  $F'_1$ , and this maps  $v$  exactly the parallelism map. It follows from Lemma 4.16 that  $\pi_1 \circ \bar{h}_1$  is  $\text{Stab}(v(F'_2))$ -invariant. Lemma 4.15 and induction assumption (1) imply

$\pi_1 \circ \bar{h}_1$  is  $Stab(v(F'_1))$ -invariant up to translation. (2) follows from (1) and (2) of Lemma 4.17. (3) follows from (3) of Lemma 4.17. (4) follows from Lemma 4.16. To see (5), we first assume  $F_3 \cap F_1 \neq \emptyset$  and take  $F_4 = F_1 \cap F_3$ , then  $\pi_1(v(F_3)) = v(F_4)$  follows from (5) of Lemma 4.17. In general, we pick a standard flat  $\tilde{F}_1$  parallel to  $F_1$  such that  $\tilde{F}_1 \cap F_3 \neq \emptyset$ . Let  $f_1 : \tilde{F}_1 \rightarrow F_1$  be the CII and let  $\tilde{\pi}_1 : F \rightarrow \tilde{F}_1$  be the CIP. Then  $f_1 \circ \tilde{\pi}_1 = \pi_1$ , which reduces the problem to the previous case. The second assertion in (5) follows from (2).  $\square$

We will construct charts for elements in  $\mathcal{F}_k(\Gamma)$  in three steps.

**Step 1:** We construct chart for a single element in  $\mathcal{F}_k(\Gamma)$ .

Pick a standard  $k$ -flat  $F \in \mathcal{F}(\Gamma)$  and vertex  $p \in F$ . Let  $F_m$  be the convex hull of all standard flats that are properly contained in  $F$ , pass through  $p$  and belong to  $\mathcal{F}(\Gamma)$ . Then  $F_m \in \mathcal{F}(\Gamma)$  by Lemma 2.18.

*Case 1:*  $F_m$  is a point. Let  $F' = \phi(F)$  and let  $h' : v(F') \rightarrow \mathbb{Z}^{v(L'(\Delta(F')))}$  be the chart for  $F'$ . Define  $h = h' \circ \phi : v(F) \rightarrow \mathbb{Z}^{v(L'(\Delta(F')))}$ . Let  $(h_1, h_2, \dots, h_k)$  be components of  $h$ . Denote the identity element in  $\mathbb{Z}^{v(L'(\Delta(F')))}$  by  $\mathbf{0}$  and let  $r = |h^{-1}(\mathbf{0})|$ . Since elements in  $h^{-1}(\mathbf{0})$  are representatives of the orbits of the action  $Stab(v(F')) \curvearrowright v(F)$ , there is a natural map  $v(F) \rightarrow h^{-1}(\mathbf{0})$ . By post-composing this map with a bijection  $h^{-1}(\mathbf{0}) \rightarrow \{0, 1, \dots, r-1\}$ , we obtain a  $Stab(v(F'))$ -invariant map  $\chi : v(F) \rightarrow \{0, 1, \dots, r-1\}$ . Now define  $\bar{h} : v(F) \rightarrow \mathbb{Z}^{v(L'(\Delta(F')))}$  by sending  $x \in v(F)$  to  $(rh_1(x) + \chi(x), h_2(x), \dots, h_k(x))$ , then  $\bar{h}$  is a bijection and we have the following commutative diagram:

$$\begin{array}{ccccc}
 & v(F) & \xrightarrow{\phi} & v(F') & \\
 \bar{h} \swarrow & \downarrow \bar{h} & \searrow h & \downarrow h' & \\
 \mathbb{Z}^{v(L'(\Delta(F))}) & \xrightarrow{s'} & \mathbb{Z}^{v(L'(\Delta(F'))}) & \xrightarrow{\phi'} & \mathbb{Z}^{v(L'(\Delta(F'))})
 \end{array}$$

Here  $\phi'$  is the map induced by  $\phi$ ,  $s'$  is the bijection induced by  $s : \Delta(F) \rightarrow \Delta(F')$  and  $\bar{h} = s'^{-1} \circ \tilde{h}$ . By construction,  $\bar{h}$  is  $Stab(v(F'))$ -invariant up to translation and satisfy condition (2) in the induction hypothesis. We choose  $\bar{h}$  to be the chart for  $F$ , which is trivially compatible with the charts already defined.

*Case 2:*  $p \subsetneq F_m \subsetneq F$ . Let  $F'$  and  $h$  be as before and let  $F'_m = \phi(F_m)$ . Suppose  $F_c$  (or  $F'_c$ ) is a standard flat which is the orthogonal complement of  $F_m$  (or  $F'_m$ ) in  $F$  (or  $F'$ ). Then we have the following commuting diagram:

$$\begin{array}{ccc}
 v(F) & \xrightarrow{h} & \mathbb{Z}^{v(L'(\Delta(F')))} \\
 \downarrow \pi & & \downarrow \pi'_c \\
 v(F_c) & \xrightarrow{h_c} & \mathbb{Z}^{v(L'(\Delta(F'_c)))}
 \end{array}$$

Here  $\pi$  and  $\pi'_c$  are the natural projections. Note that  $h$  maps fibres of  $\pi$  to fibres of  $\pi'$ , which induces  $h_c$ .  $Stab(v(F'_c))$  permutes the fibres of  $\pi$ , which induces an action  $Stab(v(F'_c)) \curvearrowright v(F_c)$ . As in case 1, we can obtain from  $h_c$  a bijection  $\bar{h}_c : v(F_c) \rightarrow \mathbb{Z}^{v(L'(\Delta(F'_c)))}$  which is  $Stab(v(F'_c))$ -invariant up to translation. Then  $\bar{h}_c \circ \pi$  is  $Stab(v(F'_m))$ -invariant (since  $Stab(v(F'_m))$  stabilizes each fibre of  $\pi$  by (4.5)) and  $Stab(v(F'_c))$ -invariant up to translation.



Let  $\bar{h}_m$  be the composition  $v(F) \rightarrow v(F_m) \rightarrow \mathbb{Z}^{v(L(\Delta(F_m)))}$  of a CIP with a chart map. Then  $\bar{h}_m$  is  $\text{Stab}(v(F'_m))$ -invariant up to translation and  $\text{Stab}(v(F'_c))$ -invariant by (1) of Lemma 4.18. Now we identify  $\mathbb{Z}^{v(L(\Delta(F_m)))}$  and  $\mathbb{Z}^{v(L(\Delta(F_c)))}$  as subgroups of  $\mathbb{Z}^{v(L(\Delta(F)))}$  and define  $\bar{h} : v(F) \rightarrow \mathbb{Z}^{v(L(\Delta(F)))}$  by  $\bar{h} = \bar{h}_c \circ \pi + \bar{h}_m$ . It is clear that the bijection  $\bar{h}$  is  $\text{Stab}(v(F'))$ -invariant up to translation. We choose  $\bar{h}$  to be the chart for  $F$  and the compatibility follows from our construction.

Let  $s_0$  and  $\varphi$  be as in (2) of the induction assumption. We claim cosets of  $\mathbb{Z}^w$  are mapped to cosets of  $\mathbb{Z}^{s_0(w)}$  under  $\varphi$  for any  $w \in v(L(\Delta(F)))$ . If  $w \in v(L(\Delta(F_m)))$ , then the claim follows from Lemma 4.16 and induction assumption (2) for  $F_m$ . If  $w \in v(L(\Delta(F_c)))$ , then by Lemma 4.18 (4),  $\bar{h}_m$  maps  $\mathbb{Z}^w$ -cosets to points. The claim follows from the construction of  $\bar{h}_c$ . Thus  $\varphi$  splits into products and  $\bar{h}$  satisfies induction assumption (2).

*Case 3:*  $F_m = F$ . Then there exist standard flats  $F_1, F_2 \in \mathcal{F}_{<k}(\Gamma)$  such that  $F$  is the convex hull of  $F_1$  and  $F_2$ . Let  $F_3 = F_1 \cap F_2$ . Suppose  $F' = \phi(F)$  and  $F'_i = \phi(F_i)$ . Take  $\bar{h}_i$  to be the charts for  $F_i$  for  $1 \leq i \leq 3$  and take  $\bar{\pi}_i : F \rightarrow F_i$  to be the CIP for  $1 \leq i \leq 3$ . Define  $\bar{h} : F \rightarrow \mathbb{Z}^{v(L(\Delta(F)))}$  by  $\bar{h} = \bar{h}_1 \circ \bar{\pi}_1 + \bar{h}_2 \circ \bar{\pi}_2 - \bar{h}_3 \circ \bar{\pi}_3$ . Then by (1) of Lemma 4.18,  $\bar{h}$  is  $\text{Stab}(v(F'))$ -invariant up to translation.

$\bar{h}$  is a bijection. It suffices to show for any standard flat  $\tilde{F}_3 \subset F$  parallel to  $F_3$ ,  $\bar{h}$  maps  $\tilde{F}_3$  bijectively to a coset of  $\mathbb{Z}^{v(L(\Delta(F_3)))}$ . Note that by Lemma 4.16, if we change the standard flats  $F_1$  and  $F_2$  in the definition of  $\bar{h}$  to some other flats parallel to them, then  $\bar{h}$  would differ by translation, thus we can assume  $\tilde{F}_3 = F_3$ . But  $\bar{h}$  restricted to  $F_3$  is of form  $\bar{h}_1 + \bar{h}_2 - \bar{h}_3$ , so what we need to prove is implied by the compatibility condition.

$\bar{h}$  is compatible with other charts. Let  $F_4 \subset F$  be an element in  $\mathcal{F}_{<k}(\Gamma)$  and let  $\bar{h}_4$  be its chart. We can assume  $F_i \cap F_4 \neq \emptyset$  by moving  $F_1$  and  $F_2$  appropriately as before. For  $1 \leq i \leq 3$ , let  $F_{4i} = F_4 \cap F_i$ , let  $\bar{\pi}_{4i} : F \rightarrow F_{4i}$  be the CIP and let  $\bar{h}_{4i}$  be the chart for  $F_{4i}$ . By (5) of Lemma 4.18,  $\pi_i(F_4) = F_{4i}$  for  $1 \leq i \leq 3$  and  $\bar{h} = \bar{h}_1 \circ \bar{\pi}_{41} + \bar{h}_2 \circ \bar{\pi}_{42} - \bar{h}_3 \circ \bar{\pi}_{43}$  when restricted on  $F_4$ . On the other hand, (3) of Lemma 4.18 and the compatibility condition imply  $\bar{h}_4 = \bar{h}_{41} \circ \bar{\pi}_{41} + \bar{h}_{42} \circ \bar{\pi}_{42} - \bar{h}_{43} \circ \bar{\pi}_{43}$  up to translation. Now the compatibility of  $\bar{h}_4$  and  $\bar{h}$  follows from the compatibility of  $\bar{h}_i$  and  $\bar{h}_{4i}$  ( $1 \leq i \leq 3$ ).

$\bar{h}$  satisfies induction assumption (2). It suffices to show  $\bar{h}$  restricted on each  $\mathbb{Z}$  coset has the desired property. Let  $w \in v(L(\Delta(F)))$  and let  $K$  be a  $\mathbb{Z}^w$  coset. Then  $K$  is contained in either a  $\mathbb{Z}^{v(L(\Delta(F_1)))}$  coset or a  $\mathbb{Z}^{v(L(\Delta(F_2)))}$  coset. Since  $\bar{h}$  is compatible with other charts, there exists standard flat  $F_5 \subset F$  parallel to either  $F_1$  or  $F_2$  such that  $K \subset \bar{h}(v(F_5))$ . Moreover, if  $\bar{h}_5$  is the chart for  $F_5$ , then  $\bar{h}_5 \circ \bar{h}^{-1}(K)$  is again a  $\mathbb{Z}^w$  coset. By applying the induction assumption to  $\bar{h}_5$ , we know  $\bar{h}|_K$  has the desired behaviour.

The above properties of  $\bar{h}$  is enough for us to establish (3) of Lemma 4.17 and (3) of Lemma 4.18 for the chart of  $F$ . Thus if we choose different  $F_1$  and  $F_2$  in the definition of  $\bar{h}$ , then the resulting chart remains the same up to translation.

**Step 2:** We construct charts for flats which have the same  $\phi$ -image as  $F$ .

We define a graph  $\Lambda(F)$ . Its vertices are in 1-1 correspondence to standard flats that have the same  $\phi$ -image as  $F$  and two vertices are joined by an edge if and only if the corresponding flats are *boltd*. The next step is to define charts for vertices of  $\Lambda(F)$  such that induction assumption (3) is satisfied. Two parallel elements  $H_1, H_2 \in \mathcal{F}(\Gamma)$  are *boltd* if there exists  $H \in \mathcal{F}(\Gamma)$  such that for  $i = 1, 2$ ,

$H \cap H_i \neq \emptyset$ ,  $H \cap H_i \subsetneq H_i$  and the convex hull of  $H$  and  $H_1$  is a flat.  $H$  is called a  $(H_1, H_2)$ -bolt, we will omit  $(H_1, H_2)$  when they are clear.

If  $\Lambda(F)$  is disconnected, we pick a representative in each connected component that does not contain  $F$ . Let  $F_0$  be one of such representatives and we build a chart  $\bar{h}_0$  for  $F_0$  as before. Pick  $y \in v(F')$ , we claim  $(\phi \circ \bar{h}_0^{-1})^{-1}(y)$  and  $(\phi \circ \bar{h}^{-1})^{-1}(y)$  are boxes of the same size. Note that  $F$  and  $F_0$  must be in the same case of step 1, so the claim follows from (3) of Lemma 4.4 in case 1 and 2. In case 3, since up to translation, the definition of  $\bar{h}_0$  does not depend on the choice of the pair of stable flats in  $F_0$ , we can choose them such that they are parallel to  $F_1, F_2 \subset F$  respectively and the claim follows. We deduce from the claim that there exists a unique identification  $f : v(F) \rightarrow v(F_0)$  characterized by (3a) and (3b) of the induction assumption. (3c) is trivially true for  $f$  and  $f$  is  $\text{Stab}(v(F'))$ -equivariant since  $\bar{h}$  and  $\bar{h}_0$  are  $\text{Stab}(v(F'))$ -invariant up to translation. Now every representative has been identified with  $F$  and induced identification between representatives obviously satisfies induction assumption (3).

It remains to define charts for flats inside one connected component, so we assume  $\Lambda(F)$  is connected.

**Lemma 4.19.** *There is a collection of bijections between each pair of flats in  $\Lambda(F)$ , which is also called CII's, such that*

- (1) *These CII's are compatible under compositions.*
- (2) *Each CII is  $\text{Stab}(v(F'))$ -equivariant and satisfies induction assumption (3a) and (3c).*
- (3) *Let  $f : H_1 \rightarrow H_2$  be a CII between flats  $H_1$  and  $H_2$  in  $\Lambda(F)$ . Suppose  $S_1 \in \mathcal{F}_{<k}(\Gamma)$  be a standard flat in  $H_1$ . Then there exists  $S_2 \in \mathcal{F}_{<k}(\Gamma)$  parallel to  $S_1$  such that  $f(v(S_1)) = v(S_2)$  and  $f|_{v(S_1)}$  is the CII between  $v(S_1)$  and  $v(S_2)$ .*

For any flat  $H$  in  $\Lambda(F)$ , we define the chart of  $H$  to be the composition of the CII between  $H$  and  $F$ , and the chart map of  $F$ . This chart satisfies induction assumption (2) since  $F$  also satisfies this condition and the CII satisfies (3a). Recall that the chart of  $F$  is compatible with the charts for flats in  $\mathcal{F}_{<k}(\Gamma)$ , so is the chart of  $H$  by Lemma 4.19 (3). Moreover, this chart is  $\text{Stab}(v(F'))$  invariant up to translation by Lemma 4.19 (2). Under such definition of charts, the CII between two flats in  $\Lambda(F)$  satisfies induction assumption (3b) by Lemma 4.19 (1).

*Proof of Lemma 4.19.* In case 1, we define the CII between any two flats in  $\Lambda(F)$  to be the map induced by parallelism, then (1) of Lemma 4.19 is true. Let  $F_1$  and  $F_2$  be a pair of bolted flats and let  $H$  be a bolt. Suppose  $f_{12} : v(F_1) \rightarrow v(F_2)$  is the CII. Then for  $i = 1, 2$ ,  $H \cap F_i$  must be one point and we denote it by  $p_i$ . It is clear that  $f_{12}(p_1) = p_2$ , thus induction assumption (3c) follows. Moreover,

$$\begin{aligned} \phi(p_1) &= \phi(v(H) \cap v(F_1)) = \phi(v(H)) \cap \phi(v(F_1)) = \phi(v(H)) \cap \phi(v(F_2)) \\ &= \phi(v(H) \cap v(F_2)) = \phi(p_2) = \phi \circ f_{12}(p_1). \end{aligned}$$

The second and fourth equality follows from Lemma 4.4 (2). Thus (3a) is true for bolted pair of flats. By moving the bolt  $H$  around using the action of  $\text{Stab}(F')$ , we know  $f_{12}$  is  $\text{Stab}(F')$ -equivariant. The connectivity of  $\Lambda(F)$  implies that (3a) and the equivariance are true for all pair of flats in  $\Lambda(F)$ .

Let  $H$  be a standard flat. An  $H$ -fibre is a standard flat parallel to  $H$ . Let  $H_1, H_2 \in \mathcal{F}(\Gamma)$  be parallel elements that contain  $H$ -fibres and let  $p : v(H_1) \rightarrow v(H_2)$

be the map induced by parallelism. We say a bijection  $f : v(H_1) \rightarrow v(H_2)$  is *parallel mod  $H$ -fibres* if  $f(v(H')) = p(v(H'))$  for any  $H$ -fibre  $H'$ . For standard flat  $S_i \in H_i$ , we will write  $f(S_1) = S_2$  if  $f(v(S_1)) = v(S_2)$ .

In case 2, for flats  $H_1$  and  $H_2$  in  $\Lambda(F)$ , we define the CII  $f : v(H_1) \rightarrow v(H_2)$  such that  $f$  is parallel mod  $F_m$ -fibres and for each  $F_m$ -fibre  $T \subset H_1$ ,  $f|_{v(T)}$  is the CII between  $v(T)$  and  $f(v(T))$ . (1) follows from parallelism and Lemma 4.17 (1) for CII's between  $F_m$ -fibres. Let  $F_1, F_2, f_{12}$  and  $H$  be as in case 1. Then for  $i = 1, 2$ , there exist  $F_m$ -fibres  $F_{im} \subset F_i$  such that  $F_i \cap H \subset F_{im}$ . Note that  $f_{12}(v(F_{1m})) = v(F_{2m})$  and  $H$  is also a bolt for  $F_{1m}$  and  $F_{2m}$  when  $F_1 \cap H \subsetneq F_{1m}$ , thus induction assumption (3c) follows. By Lemma 2.18, we can assume  $H \cap F_i$  is actually a  $F_m$ -fibre for  $i = 1, 2$ , then the argument in the previous case implies that the image of any  $F_m$ -fiber in  $F_1$  under  $\phi$  and  $\phi \circ f_{12}$  are the same. Then (3a) follows since we already know it is true for CII's between  $F_m$ -fibres. The  $\text{Stab}(F')$ -equivariance follows by applying Lemma 4.17 (4) to CII's between  $F_m$ -fibres. Note that any element of  $\mathcal{F}_{<k}(\Gamma)$  that lies in  $F_1$  must stay inside a  $F_m$ -fibre, then Lemma 4.19 (3) follows from Lemma 4.17 (2).

In case 3, let  $H_1$  and  $H_2$  be a bolted pair in  $\Lambda(F)$ . Pick a vertex  $p_0 \in H_1$  and let  $H$  be the intersection of all  $(H_1, H_2)$ -bolts that contains  $p_0$ . Then  $H$  is also a bolt. We define the CII  $f : v(H_1) \rightarrow v(H_2)$  as in case 2 with  $F_m$ -fibres replaced by  $H \cap H_1$ -fibres. The induction assumption (3c) for  $f$  follows from the minimality of  $H$  and we can prove (3a) and the  $\text{Stab}(v(F'))$ -equivariance as before.

Now we prove Lemma 4.19 (3) for  $f$ . It is clear if  $S_1$  stays inside a  $H \cap H_1$ -fibre. In general, pick a  $H \cap H_1$ -fibre  $T_1$  such that  $T_1 \cap S_1 = S_{11} \neq \emptyset$  and a standard flat  $S_{12}$  which is an orthogonal complement of  $S_{11}$  in  $S_1$ . Since  $f$  is parallel mod  $T_1$ -fibres,  $f(v(S_1))$  belongs to a  $(T_1 \times S_{12})$ -fibre  $R$ . Suppose  $S_{21} = f(S_{11})$  and  $T_2 = f(T_1)$ . Let  $\pi_i : H_i \rightarrow T_i$  be the CIP for  $i = 1, 2$ . Then  $\pi_1(v(S_1)) = S_{11}$  by Lemma 4.18 (5), hence  $\pi_2(f(v(S_1))) = S_{21}$  by Lemma 4.17 (1). But every two  $T_1$ -fibres in  $R$  are bolted by  $S_1$ -fibres, then the CII between these two  $T_1$ -fibres is parallel mod  $S_{11}$ -fibres, which implies  $f(v(S_1))$  actually stays inside a  $(S_{11} \times S_{12})$ -fibre. To see the second part of Lemma 4.19 (3), note that  $S_1$  and  $S_2$  are bolted by  $H \cap H_1$ -fibres, then the CII between them is parallel mod  $S_{11}$ -fibres by induction assumption (3c). Thus the CII coincides with  $f$  by Lemma 4.17 (2).

For arbitrary pair  $H_1$  and  $H_2$  in  $\Lambda(F)$ , we choose an edge path in  $\Lambda(F)$  connecting  $H_1$  and  $H_2$ , which would induce a CII from  $H_1$  to  $H_2$ . This CII will automatically satisfies  $\text{Stab}(v(F'))$ -equivariance, induction assumption (3a) and Lemma 4.19 (3), since these properties are true under compositions. For this CII to be well-defined, we need to show every edge loop in  $\Lambda(F)$  induces the identity map. Let  $F$  be a base point in the edge loop and let  $f : v(F) \rightarrow v(F)$  be the bijection induced by the edge loop. Pick  $F_1, F_2 \in \mathcal{F}_{<k}$  inside  $F$  such that their convex hull is  $F$ , then it follows from (3c) that for  $i = 1, 2$ , every CII between two  $F_i$ -fibres in  $F$  is parallel mod  $F_1 \cap F_2$ -fibres. We first assume  $F_1 \cap F_2$  is a point. By previous discussion,  $f$  maps  $F_i$ -fibre to  $F_i$ -fibre, thus  $f$  splits into product  $f = f_1 \times f_2$  where  $f_i : F_i \rightarrow F_i$  are bijections. Moreover, if  $g : f(v(F_1)) \rightarrow v(F_1)$  is the CII, then  $g \circ f|_{v(F_1)} = \text{Id}$  by (1) of Lemma 4.17, thus  $f|_{v(F_1)}$  is induced by parallelism and  $f_2 = \text{Id}$ . Similarly we can prove  $f_1 = \text{Id}$ , thus  $f = \text{Id}$ . In general, we can run the same argument mod  $F_1 \cap F_2$ -fibres to show that  $f$  sends every  $F_1 \cap F_2$ -fibre to itself, then  $f = \text{Id}$  follows by applying Lemma 4.17 (1) to  $F_1 \cap F_2$ -fibres.  $\square$

### Step 3:

Let  $H$  be an element in  $\mathcal{F}_k(\Gamma)$  such that  $\phi(H)$  is in the  $G(\Gamma')$ -orbit of  $F'$ . Note that this is equivalent to  $L'(\Delta(\phi(H))) = L'(\Delta(F'))$ . Pick  $g' \in G(\Gamma')$  with  $\bar{\phi}_{g'}(\phi(H)) = F'$ , then  $\phi_{g'}(H)$  is an element in  $\Lambda(F)$ . We define the chart of  $H$  to be the composition of the chart map of  $\phi_{g'}(H)$  and  $\phi_{g'}$ . If we choose a different  $g'$ , the resulting chart would differ by a translation, since  $\bar{h}$  is  $\text{Stab}(v(F'))$ -invariant up to translation. By (4.3), this chart satisfies induction assumption (2). Moreover, it is compatible with charts of elements in  $\mathcal{F}_{<k}(\Gamma)$ , since these charts are  $G(\Gamma')$ -invariant up to translations, and they are compatible with charts of flats in  $\Lambda(F)$  by the previous step.

By now we have defined a  $G(\Gamma')$ -invariant (up to translations) collection of charts for flats that are  $G(\Gamma')$  orbits of flats in  $\Lambda(F)$ . This collection corresponds to a stable clique of  $k$  vertices in  $\Gamma'$ , namely the 1-skeleton of  $L'(\Delta(F'))$ . For each stable  $k$ -clique in  $\Gamma'$ , we run the same argument to define charts for the corresponding collection of  $k$ -flats in  $G(\Gamma)$ . This gives rise to charts defined for all elements in  $\mathcal{F}_k(\Gamma)$  that satisfies all the requirements, hence finishes the induction step. In summary, we have constructed a  $G(\Gamma')$ -invariant (up to translations)  $L$ -atlas  $\bar{\mathcal{A}}_L$  such that induction assumption (2) is true for all charts in this atlas.

**Theorem 4.20.** *Let  $G(\Gamma)$  be a group of weak type I. Then the following are equivalent.*

- (1)  $G(\Gamma')$  is quasi-isometric to  $G(\Gamma)$ .
- (2)  $G(\Gamma')$  is isomorphic to a finite index subgroup of  $G(\Gamma)$ .
- (3)  $G(\Gamma')$  is isomorphic to a special subgroup (Section 2.4) of  $G(\Gamma)$ .

*Proof of Theorem 4.20.* (3)  $\Rightarrow$  (1) and (2)  $\Rightarrow$  (1) are trivial. Now we look at (1)  $\Rightarrow$  (2). By Theorem 2.16, we can assume  $G(\Gamma)$  has trivial centre. Let  $\mathcal{A}_{ref}$  be the reference atlas for  $G(\Gamma)$ . By Lemma 4.9, there exists simplicial isomorphism  $r : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  such that  $r^*(\bar{\mathcal{A}}_L) \stackrel{e}{=} \mathcal{A}_{ref}$ . The  $G(\Gamma')$ -invariance of  $\bar{\mathcal{A}}_L$  implies  $(r^{-1} \circ s_{g'} \circ r)^*(\mathcal{A}_{ref}) \stackrel{e}{=} \mathcal{A}_{ref}$ , thus  $\phi_r^{-1} \circ \phi_{g'} \circ \phi_r$  is a left translation of  $G(\Gamma)$  (here  $\phi_r : G(\Gamma) \rightarrow G(\Gamma)$  is the map induced by  $r$ ). Hence we have obtained a faithful action of  $G(\Gamma')$  on  $G(\Gamma)$  with finite many orbits. Thus (2) follows.

(1)  $\Rightarrow$  (3). Since the atlas  $\bar{\mathcal{A}}_L$  satisfies induction assumption (2),  $\phi \circ \phi_r$  sends standard flats to standard flats. Moreover, it extends to a cubical map  $X(\Gamma) \rightarrow X(\Gamma')$ . The inverse image of a vertex under this cubical map is a compact convex subcomplex of  $X(\Gamma)$ . Then [Hua14a, Section 6] implies  $G(\Gamma')$  is isomorphic to the special subgroup associated with this convex subcomplex.  $\square$

Though a finite index RAAG subgroup of  $G(\Gamma)$  is isomorphic to a special subgroup, it may not be a special subgroup. However, this is true under the strong condition that  $\text{Out}(G(\Gamma))$  is finite [Hua14a, Section 6].

The following is a consequence of Theorem 4.20 and [Hua14a, Section 6.3].

**Corollary 4.21.** *Let  $G(\Gamma)$  be a group of weak type I. Then there is an algorithm to determine whether  $G(\Gamma')$  and  $G(\Gamma)$  are quasi-isometric.*

## 5. SHUFFLE TIRES AND BRANCHES

**5.1. Prime right-angled Artin group.** From now on, we assume  $G(\Gamma)$  is of type II. We also label and orient edges of  $X(\Gamma)$  in a  $G(\Gamma)$ -invariant way (see Section 2.1).

Let  $q : G(\Gamma) \rightarrow G(\Gamma')$  be a quasi-isometry and let  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$  be the induced simplicial isomorphism. Pick vertex  $v \in \mathcal{P}(\Gamma)$ , then  $q_*$  induces a 1-1

correspondence between  $v$ -branches in  $\mathcal{P}(\Gamma)$  and  $q_*(v)$ -branches in  $\mathcal{P}(\Gamma')$ . This correspondence is the starting point to understand the quasi-isometry  $q$ .

**Definition 5.1.** Let  $v \in \mathcal{P}(\Gamma)$  be a vertex. Two  $v$ -branches  $B_1$  and  $B_2$  are *quasi-isometrically indistinguishable* (QII) if there exist a quasi-isometry  $f : X(\Gamma) \rightarrow X(\Gamma)$  and an induced simplicial isomorphism  $f_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  such that

- (1)  $f_*$  fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (B_1 \cup B_2)$ .
- (2)  $f_*(B_1) = B_2$  and  $f_*(B_2) = B_1$ .

Such  $f$  or  $f_*$  will be called an *elementary permutation*.

It follows immediately from the definition that  $q_*$  sends QII  $v$ -branches to QII  $q_*(v)$ -branches.

**Lemma 5.2.** *Pick a  $v$ -tier  $T$ , then for any  $v$ -branch  $B$ , there exists a  $v$ -branch  $B' \subset T$  such that  $B'$  and  $B$  are QII.*

*Proof.* Pick standard geodesic  $l \subset X(\Gamma)$  such that  $\Delta(l) = v$  and suppose  $\pi_{\Delta(l)}(B) = x$  and  $\pi_{\Delta(l)}(T) = x'$  ( $\pi_{\Delta(l)}$  is the map in Lemma 3.1). Recall that we have an action  $G(\Gamma) \curvearrowright X(\Gamma)$ , let  $\alpha \in G(\Gamma)$  be the element such that  $\alpha$  acts by translation along  $l$  and  $\alpha(x) = x'$ . It is clear that  $\alpha$  induces a simplicial isomorphism  $\alpha_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$ , moreover,  $\alpha_*$  fixes every point in  $St(v)$ . Define  $B' = \alpha_*(B)$  and let  $L$  and  $L'$  be the components of  $X(\Gamma) \setminus P_v$  corresponding to  $B$  and  $B'$  respectively (Lemma 3.26). Then  $\alpha(L) = L'$ . Now we consider the following map  $q : X(\Gamma) \rightarrow X(\Gamma)$  defined by

$$q(z) = \begin{cases} z & \text{if } z \in X(\Gamma) \setminus (L \cup L') \\ \alpha(z) & \text{if } z \in L \\ \alpha^{-1}(z) & \text{if } z \in L' \end{cases}$$

It is easy to see  $q$  is a quasi-isometry and Lemma 3.26 implies that  $q_*$  satisfies the conditions in Definition 5.1, so  $B$  and  $B'$  are QII.  $\square$

Let  $\bar{v} = \pi(v)$ . It follows from Corollary 3.14 and Lemma 3.26 that we can assign a component  $C$  of  $F(\Gamma) \setminus St(\bar{v})$  for every  $v$ -branch  $B$ , and we will denote  $C = \Pi(B)$  in this case.

Two components  $C_1$  and  $C_2$  of  $F(\Gamma) \setminus St(\bar{v})$  are *quasi-isometrically indistinguishable* (QII) if there exist  $v$ -branches  $B_1$  and  $B_2$  which are QII such that  $\Pi(B_i) = C_i$  for  $i = 1, 2$ . Note that this definition actually does not depend on the choice of  $B_1$  and  $B_2$ , namely, if  $C_1$  and  $C_2$  are QII, then for any pair  $B'_1$  and  $B'_2$  such that  $\partial B'_1 = \partial B'_2$  and  $\Pi(B'_i) = C_i$ , they are QII. Actually, the case  $\partial B'_1 = \partial B_1$  follows from Lemma 5.2. If  $\partial B'_1 \neq \partial B_1$ , let  $K$  and  $K'$  be the standard subcomplexes in  $P_v$  such that  $\Delta(K') = \partial B'_1$  and  $\Delta(K) = \partial B_1$ . Note that their supports satisfy  $\Gamma_K = \Gamma_{K'}$ , then there exists  $\alpha \in G(\Gamma)$  such that its action on  $G(\Gamma)$  satisfying (1)  $\alpha(K) = K'$ ; (2)  $\alpha(P_v) = P_v$ . Since  $\alpha$  is label-preserving,  $\Pi(\alpha_*(B_1)) = \Pi(B_1)$ . Moreover,  $\alpha_*(v) = v$ , so  $\alpha_*(B_1)$  and  $\alpha_*(B_2)$  is a pair of QII  $v$ -branches with  $\partial(\alpha_*(B_1)) = \partial B'_1$  and we have reduced to the previous case.

Let  $\{\mathcal{C}_i\}_{i=1}^k$  be the collection of QII classes in  $F(\Gamma) \setminus St(\bar{v})$ . We associate  $\bar{v}$  with a  $k$ -tuple of positive integers  $(n_1, n_2, \dots, n_k)$ , where each  $n_i$  is the number of components of  $F(\Gamma) \setminus St(\bar{v})$  in  $\mathcal{C}_i$ . The vertex  $v$  is *prime* if  $\gcd(n_1, n_2, \dots, n_k) = 1$ .

It follows from (1) of Corollary 3.14 that if  $C_1$  and  $C_2$  are QII, then  $\partial C_1 = \partial C_2$ , so every QII class  $\mathcal{C}_i$  has a well-defined boundary, which will be denoted by  $\partial \mathcal{C}_i$ .

**Definition 5.3.** A right-angled Artin group  $G(\Gamma)$  is *prime* if and only if  $F(\Gamma)$  is of type II and all vertices of  $F(\Gamma)$  are prime.

Let  $\bar{v}' = \pi(q_*(v))$ . Then  $q_*$  induces a bijection between QII classes of  $F(\Gamma) \setminus St(\bar{v})$  and QII classes of  $F(\Gamma') \setminus St(\bar{v}')$ . For  $1 \leq i \leq k$ , let  $\mathcal{C}'_i$  be the class corresponding to  $\mathcal{C}_i$  under  $q_*$ , and let  $n'_i$  be the number of components of  $F(\Gamma') \setminus St(\bar{v}')$  in  $\mathcal{C}'_i$ . Note that  $(n'_1, n'_2, \dots, n'_k)$  is the tuple associated with  $\bar{v}'$ .

**Lemma 5.4.** *There exists a positive rational number  $r$  such that  $n'_i = rn_i$  for  $1 \leq i \leq k$ . In particular, if  $G(\Gamma)$  is prime, then  $r$  is an integer.*

Such  $r$  will be called the *stretch factor* of  $q_*$  at  $v$ .

*Proof.* Let  $l \subset X(\Gamma)$  and  $l' \subset X(\Gamma')$  be standard geodesics such that  $\Delta(l) = v$  and  $\Delta(l') = q_*(v)$ . Recall that the vertex set  $v(l)$  has a natural ordering induced from the orientation of edges in  $X(\Gamma)$ . We identify  $v(l)$  with  $\mathbb{Z}$  in an order-preserving way. Pick  $\mathcal{C}_i$  and pick a  $v$ -branch  $B$  such that  $\Pi(B) \in \mathcal{C}_i$ . Let  $\{B_j\}_{j \in J}$  be the collection of  $v$ -branches such that  $B_j$  and  $B$  are QII. Then by (1) and (2) of Corollary 3.14, there are exactly  $n_i$  elements of  $\{B_j\}_{j \in J}$  in a given  $v$ -tier. Pick an total order on elements in  $\mathcal{C}_i$  and define an total order on  $J$  by  $j_1 < j_2$  if and only if  $\pi_v(B_{j_1}) < \pi_v(B_{j_2})$  ( $\pi_v$  is the map in Lemma 3.1) or  $\pi_v(B_{j_1}) = \pi_v(B_{j_2})$  and  $\Pi(B_{j_1}) < \Pi(B_{j_2})$ . We identify  $J$  with  $\mathbb{Z}$  in an order-preserving way, then there is a natural map  $g_i : J \rightarrow v(l)$  induced by  $\pi_v$ . Note that  $g_i(a) = \lfloor a/n_i \rfloor$  up to translation.

Let  $\{B'_k\}_{k \in K}$  be the collection of  $q_*(v)$ -branches such that  $B'_k$  and  $q_*(B)$  are QII. Then  $\Pi(B'_k) \in \mathcal{C}'_i$  and  $q_*$  induces a bijection  $f_i : J \rightarrow K$ . We identify  $v(l')$  with  $\mathbb{Z}$  and  $K$  with  $\mathbb{Z}$  in the same way as before and let  $g'_i : K \rightarrow v(l')$  be the natural map given by  $g'_i(a) = \lfloor a/n'_i \rfloor$ .

We define another map  $h_i : v(l) \rightarrow v(l')$  as follows. For  $x \in v(l)$ , pick a  $B_j$  such that  $\pi_v(B_j) = x$  and define  $h_i(x) = \pi_{\Delta(l')}(q_*(B_j))$ . Up to bounded distance, the definition of  $h_i$  is independent of the choice of  $B_j$ . We claim  $h_i$  is a quasi-isometry. Pick  $B_{j_1}, B_{j_2}$  in  $\{B_j\}_{j \in J}$ . For  $m = 1, 2$ , let  $L_{j_m}$  be the subset of  $X(\Gamma)$  as in Lemma 3.26 such that  $\Delta(\bar{L}_{j_m}) = B_{j_m}$ . Then  $d(L_{j_1}, L_{j_2}) = d(\pi_v(B_{j_1}), \pi_v(B_{j_2}))$ . Now it follows from (3) and (5) of Lemma 3.26 that  $h_i$  is a quasi-isometry. Note that the following diagram commutes up to bounded distance:

$$\begin{array}{ccc} J & \xrightarrow{f_i} & K \\ \downarrow g_i & & \downarrow g'_i \\ v(l) & \xrightarrow{h_i} & v(l') \end{array}$$

thus  $f_i$  is a bijective quasi-isometry from  $\mathbb{Z}$  to  $\mathbb{Z}$ , hence  $f_i$  is bounded distance from an isometry and  $h_i(x) = (n_i/n'_i)x + b$  up to bounded distant ( $b$  is some constant). Now we pick a different QII class  $\mathcal{C}_{i'}$  and define  $h_{i'}$  in similar way, then  $h_i = h_{i'}$  up to bounded distant, but we also have  $h_{i'}(x) = (n_{i'}/n'_{i'})x + b'$  up to bounded distance, so  $n_i/n'_i = n_{i'}/n'_{i'}$ .  $\square$

Since  $q_*$  does not necessarily map  $v$ -tier to  $q_*(v)$ -tier, we want to modify  $q_*$  by post-composing  $q_*$  with an appropriate permutation of  $q_*(v)$ -branches such that  $v$ -tier is sent to  $q_*(v)$ -tier. It is easy to do this for a single vertex  $v$  when  $G(\Gamma)$  is prime by Lemma 5.4, but in general we need to deal with more than one vertices simultaneously, so it is necessary to figure out how to deal with each vertex independently.

**Lemma 5.5.** *Assume  $d(v_1, v_2) \geq 2$  and let  $B$  be any  $v_1$ -branch such that  $v_2 \notin B$ . Then  $B$  and  $v_1$  are in the same  $v_2$ -branch, in particular, all such  $B$  are in the same  $v_2$ -branch.*

*Proof.* Note that  $B \cap St(v_2) = \emptyset$  (otherwise  $v_2 \in B$ ) and  $\partial B \not\subseteq lk(v_1) \cap lk(v_2) = St(v_1) \cap St(v_2)$  ((5) of Corollary 3.14), so there exists vertex  $w' \in \partial B$  with  $w' \notin St(v_2)$ , hence  $B$  can be connected to  $v_1$  via  $w'$  outside  $St(v_2)$ .  $\square$

Now we introduce an auxiliary notion. Pick vertex  $w, v \in \mathcal{P}(\Gamma)$  and a  $v$ -peripheral complex  $K \subset \mathcal{P}(\Gamma)$ . The pair  $(v, K)$  is  $w$ -non-crossing if  $d(v, w) = 1$  and  $w \notin K$ . In this case,  $B \cap St(w) = \emptyset$  for any  $v$ -branch  $B$  such that  $\partial B = K$ . Moreover, for any other  $v$ -branch  $B'$  with  $\partial B' = K$ ,  $B'$  and  $B$  are in the same  $w$ -branch. To see this, note that  $K = \partial B \not\subseteq lk(v) \cap lk(w)$  by Corollary 3.14 (5), so  $K$  contains a vertex  $w' \in St(v) \setminus St(w)$  such that  $B'$  can be connected with  $B$  outside  $St(w)$  via  $w'$ .

We define a binary relation  $\leq$  on the set of  $w$ -non-crossing pairs by  $(v_1, K_1) \leq (v_2, K_2)$  if and only if there exist  $v_1$ -branch  $B_1$  with  $\partial B_1 = K_1$  and  $v_2$ -branch  $B_2$  with  $\partial B_2 = K_2$  such that  $B_1 \subset B_2$ . If  $(v_1, K_1) < (v_2, K_2)$ , then  $d(v_1, v_2) = 1$ . To see this, note that if  $v_1 = v_2$ , we must have  $B_1 = B_2$  and  $K_1 = K_2$ . Suppose  $d(v_1, v_2) = 2$ . Since  $v_2 \notin B_1$ ,  $B_1$  must belong to the  $v_2$ -branch that contains  $v_1$  by Lemma 5.5. Hence  $v_1 \in B_2$  and  $w \in \partial B_2 = K_2$ , which yields a contradiction.

Suppose  $(v_1, K_1) < (v_2, K_2)$ . Since  $B_1 \subset B_2$ , we have  $v_2 \notin \partial B_1 = K_1$ . Thus  $(v_1, K_1)$  is  $v_2$ -non-crossing and we deduce as before that then  $B'_1 \subset B_2$  for any  $v_1$ -branch  $B'_1$  with  $\partial B'_1 = K_1$ . Thus the relation  $\leq$  is transitive. If  $(v_1, K_1) \leq (v_2, K_2)$ ,  $(v_2, K_2) \leq (v_1, K_1)$  and  $(v_1, K_1) \neq (v_2, K_2)$ , then it follows from previous discussion that all  $v_1$ -branches with boundary  $= K_1$  stay inside one particular  $v_1$ -branch with boundary  $= K_1$ , which is absurd. So  $\leq$  is antisymmetric, hence is a partial order.

The reader can skip the following lemma first and come back to it when needed.

**Lemma 5.6.** *Let  $F(\Gamma)$  be of type II, pick vertex  $w \in \mathcal{P}(\Gamma)$  and let  $\{B_i\}_{i=1}^n$  be a collection of distinct  $v_i$ -branches with  $d(v_i, w) = 1$  ( $v_i = v_j$  is allowed for  $i \neq j$ ). Let  $q : X(\Gamma) \rightarrow X(\Gamma)$  be a quasi-isometry such that  $q_*$  fixes every point in  $St(v)$ . Then there exists a quasi-isometry  $q' : X(\Gamma) \rightarrow X(\Gamma)$  such that  $q'_*$  satisfies:*

- (1)  $q'_*$  fixes every point in  $St(v)$ .
- (2)  $q'_*(B) = q_*(B)$  for any  $w$ -branch  $B$ .
- (3)  $B_i$  and  $q'_*(B_i)$  are in the same  $v_i$ -tier.
- (4) If  $q_*$  fixes every point in a  $w$ -branch  $B$ , then  $q'_*$  also fixes every point in  $B$ .

*Proof.* We only need to consider the case when  $B_i \subset \mathcal{P}(\Gamma) \setminus St(w)$  for all  $i$ , otherwise  $B_i$  will contain a vertex fixed by  $q_*$  and (3) is automatic. Let  $K_i = \partial B_i$ . Then  $(v_i, K_i)$  is a  $w$ -non-crossing pair. Suppose  $(v_1, K_1)$  is a maximal element in  $\{(v_i, K_i)\}_{i=1}^n$  with respect to the order defined above and suppose  $(v_1, K_1) = (v_i, K_i)$  if and only if  $1 \leq i \leq m$ . Let  $K'_1 = q_*(K_1)$  and let  $\{A_i\}_{i \in \mathbb{Z}}$  (or  $\{A'_i\}_{i \in \mathbb{Z}}$ ) be the collection of  $v_1$ -branches with boundary  $K_1$  (or  $K'_1$ ). Then  $q_*$  induces a bijection between  $\{A_i\}_{i \in \mathbb{Z}}$  and  $\{A'_i\}_{i \in \mathbb{Z}}$ . Since  $q_*$  fixes  $v_1$ , the stretch factor of  $q_*$  at  $v_1$  is 1 by Lemma 5.4, so we can post-compose  $q_*$  with a finite sequence of elementary permutations of elements in  $\{A'_i\}_{i \in \mathbb{Z}}$  such that (3) is true for  $1 \leq i \leq m$ . Note that  $(v_1, K'_1)$  is also  $w$ -non-crossing, so  $\{A'_i\}_{i \in \mathbb{Z}}$  are in the same  $w$ -branch, hence (1) and (2) still hold.

Pick  $i_0 > m$  and let  $D_1$  and  $D_2$  be two QII  $v_{i_0}$ -branches such that  $\partial D_1 = \partial D_2 = q_*(K_{i_0})$ . Let  $f_*$  be an elementary permutation of  $D_1$  and  $D_2$ . We claim

$f_*(q_*(B_i)) = q_*(B_i)$  for  $1 \leq i \leq m$ , then the lemma follows by induction on the number of  $B_i$ . To see the claim, note that  $(v_1, K'_1) \not\leq (v_{i_0}, q_*(K_{i_0}))$  (since  $(v_1, K_1) \not\leq (v_{i_0}, K_{i_0}))$ , then for any  $v_1$ -branch  $E$  such that  $\partial E = K'_1$ ,  $E$  contains a vertex  $u \in \mathcal{P}(\Gamma) \setminus (D_1 \cup D_2)$ , otherwise we would have  $E \subset D_1$  or  $E \subset D_2$ . Recall that  $f_*(u) = u$ , so  $f_*(E) = E$ , in particular  $f_*(q_*(B_i)) = q_*(B_i)$  for  $1 \leq i \leq m$ .

Property (4) is true since we only need to consider those  $B_i$ 's that are not contained in  $w$ -branches which are fixed by  $q_*$  pointwise.  $\square$

Suppose  $G(\Gamma)$  and  $G(\Gamma')$  are prime right-angled Artin groups. Let  $q : G(\Gamma) \rightarrow G(\Gamma')$  be a quasi-isometry and  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$  be the induced simplicial isomorphism. Pick vertex  $x \in X(\Gamma)$ . Let  $K = (F(\Gamma))_x$  and  $K' = q_*(K)$ . Suppose  $\{v_i\}_{i=1}^n$  is the collection of vertices in  $\mathcal{P}(\Gamma)$  such that  $K \setminus St(v_i)$  is disconnected, then  $v_i \in K$  for all  $i$  by Remark 3.8. Let  $v'_i = q_*(v_i)$ . Then  $\{v'_i\}_{i=1}^n$  are exactly the vertices in  $\mathcal{P}(\Gamma')$  such that  $K' \setminus St(v'_i)$  is disconnected.

Recall that  $\pi : \mathcal{P}(\Gamma') \rightarrow F(\Gamma')$  is the canonical projection.

**Lemma 5.7.** *If for any vertex  $v' \in \mathcal{P}(\Gamma)$ , all vertices in  $K' \setminus St(v')$  are in the same  $v'$ -tier, then  $\pi|_{K'}$  is injective. Moreover,  $\cap_{v \in K'} P_v \neq \emptyset$ .*

*Proof.* Since  $\pi$  maps simplexes to simplexes of the same dimension, it suffices to show there does not exist vertex  $w_1, w_2 \in K'$  such that  $\pi(w_1) = \pi(w_2)$ . Suppose the contrary is true. Then  $P_{w_1} \cap P_{w_2} = \emptyset$ . Let  $h$  be a hyperplane separating  $P_{w_1}$  and  $P_{w_2}$ , and let  $l$  be a standard geodesic dual to  $h$ . Then  $\pi_{\Delta(l)}(w_1) \neq \pi_{\Delta(l)}(w_2)$  ( $\pi_{\Delta(l)}$  is the map in Lemma 3.1), hence  $w_1$  and  $w_2$  are in different  $\Delta(l)$ -tier, which yields a contradiction. The second statement follows from the Lemma 2.2 and the previous argument.  $\square$

Our next goal is to post-compose  $q_*$  with elementary permutations such that the assumption of Lemma 5.7 is true. In order to do this, we introduce an auxiliary order. Pick  $E \subset \{v'_i\}_{i=1}^n$  and denote  $\{v'_i\}_{i=1}^n \setminus E$  by  $E^c$ . We say  $E$  is *tight* if for any  $v'_i \in E^c$ ,  $E \setminus St(v'_i)$  is inside a  $v'_i$ -branch. Pick  $v'_i, v'_j \in E^c$ , we define  $v'_i <_E v'_j$  if and only if there exists  $v'_k \in E$  such that  $v'_j$  and  $v'_k$  are in different  $v'_i$ -branches. We claim that if  $E$  is tight, then  $\leq_E$  is a partial order on  $E^c$ .

From now on, we will write  $v'_j|_{v'_i} v'_k$  if  $v'_j$  and  $v'_k$  are in different  $v'_i$ -branches and write  $v'_j v'_k|_{v'_i}$  if  $v'_j$  and  $v'_k$  are in the same  $v'_i$ -branch.

**Lemma 5.8.** *Suppose  $F(\Gamma)$  is of type II. Let  $v_1, v_2, v_3$  be vertices in  $\mathcal{P}(\Gamma)$ . If  $v_1|_{v_2} v_3$ , then  $v_1 v_2|_{v_3}$  and  $v_3 v_2|_{v_1}$ .*

*Proof.* Let  $B$  be the  $v_2$ -branch that contains  $v_3$ . Since  $\mathcal{P}(\Gamma) \setminus (lk(v_1) \cap lk(v_2))$  is connected,  $\partial B \not\subseteq lk(v_1) \cap lk(v_2) = St(v_1) \cap St(v_2)$ . Then there exists vertex  $w \in \partial B$  with  $w \notin St(v_1)$ , which implies that  $v_2$  and  $v_3$  can be connected via  $w$  outside  $St(v_1)$ .  $\square$

Now we prove the claim. If  $v'_i <_E v'_j$  and  $v'_j <_E v'_i$ , then there exist  $v'_{k_1}$  and  $v'_{k_2}$  in  $E$  such that  $v'_j|_{v'_i} v'_{k_1}$  and  $v'_i|_{v'_j} v'_{k_2}$ . By Lemma 5.8, we have  $v'_{k_2} v'_j|_{v'_i}$ , so  $v'_{k_2}|_{v'_i} v'_{k_1}$ , which contradicts the tightness of  $E$ . Thus the relation  $\leq_E$  is antisymmetric. It suffices to check the transitivity. Suppose  $v'_i <_E v'_j$  and  $v'_j <_E v'_k$  for  $v'_i, v'_j, v'_k \in E^c$ , then there exist  $v'_l$  and  $v'_m$  in  $E$  such that  $v'_l|_{v'_i} v'_j$  and  $v'_m|_{v'_j} v'_k$ . Since  $v'_l \notin St(v'_j)$  and  $v'_m \notin St(v'_j)$ , then  $v'_m v'_l|_{v'_j}$ . We also have  $v'_i v'_l|_{v'_j}$  by Lemma 5.8, so  $v'_m v'_i|_{v'_j}$ . This together with  $v'_m|_{v'_j} v'_k$  imply  $v'_i|_{v'_j} v'_k$ , hence  $v'_k v'_j|_{v'_i}$ . However,  $v'_l|_{v'_i} v'_j$ , so  $v'_l|_{v'_i} v'_k$  and  $v'_i <_E v'_k$ .



If  $E$  is tight, let  $v'_i \in E^c$  be a minimal element in  $E^c$  with respect to  $\leq_E$ . Then  $E \cup \{v'_i\}$  is also tight. Let  $E_1 = \{v'_1\}$ .  $E_1$  is clearly tight, so it is possible to add a vertex in  $E_1^c$  to  $E_1$  to obtain a tight set  $E_2$ . By repeating this process for  $n-1$  times, we obtain a filtration  $E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = \{v'_i\}_{i=1}^n$  such that each  $E_i$  is tight and  $|E_{i+1}| = |E_i| + 1$ . Set  $E_0 = \emptyset$ .

Suppose we have already obtained a quasi-isometry  $q_*$  such that for every vertex  $v' \in E_i$ , vertices of  $K' \setminus St(v')$  are in the same  $v'$ -tier. Suppose  $v'_{i+1} = E_{i+1} \setminus E_i$  and let  $B'$  be the  $v'_{i+1}$ -branch that contains all points in  $E_i \setminus St(v'_{i+1})$  (if  $E_i \setminus St(v'_{i+1}) = \emptyset$ , we pick an arbitrary  $v'_{i+1}$ -branch). Let  $\{B_j\}_{j=1}^k$  be the collection of  $v_{i+1}$ -branches that have non-trivial intersection with  $K$  and let  $B'_j = q_*(B_j)$ . It is clear that  $\{B_j\}_{j=1}^k$  are in the same  $v_{i+1}$ -tier, moreover, since both  $G(\Gamma)$  and  $G(\Gamma')$  are prime, the stretch factor of  $q_*$  at  $v_{i+1}$  is 1, then there exists  $g_* : \mathcal{P}(\Gamma') \rightarrow \mathcal{P}(\Gamma)$  such that (1)  $g_*$  is a composition of finite many elementary permutations of  $v'_{i+1}$ -branches, hence  $g_*$  fixes every point in  $St(v'_{i+1})$ ; (2)  $g_*$  fixes every point in  $B'$ ; (3)  $\{g_*(B'_j)\}_{j=1}^k$  are in the same  $v'_{i+1}$ -tier. By Lemma 5.6, we can assume in addition that (4) for any  $v' \in E_i \cap St(v'_{i+1})$  and any  $v'$ -branch  $D$  such that  $D \cap K' \neq \emptyset$ ,  $g_*(D)$  and  $D$  are in the same  $v'$ -tier.

By (1) and (2),  $g_*$  fixes every point in  $E_{i+1}$ . We claim that vertices of  $g_*(K') \setminus St(v')$  are in the same  $v'$ -tier for any  $v' \in g_*(E_{i+1})$ . The case  $v' = v'_{i+1}$  follows from (3) and the case  $v' \in E_i \cap St(v'_{i+1})$  follows from (4). Let  $v' \in E_i \setminus St(v'_{i+1})$  and  $D$  be a  $v'$ -branch. If  $v'_{i+1} \notin D$ , then  $g_*(D) = D$  by (2) and Lemma 5.5; if  $v'_{i+1} \in D$ ,  $g_*(D) = D$  is still true since  $g_*$  fixes  $v'$  and  $v'_{i+1}$ . Thus  $g_*$  does not permute the  $v'$ -branches and the claim follows.

Let  $q'_* = g_* \circ q_*$ ,  $K'' = g_*(K') = q'_*(K)$ ,  $E'_i = g_*(E_i)$  and  $v''_i = g_*(v'_i)$ . Then  $\{v''_i\}_{i=1}^n$  are exactly the vertices in  $\mathcal{P}(\Gamma')$  such that  $K'' \setminus St(v''_i)$  is disconnected. And  $E'_1 \subsetneq E'_2 \subsetneq \cdots \subsetneq E'_n$  is a tight filtration of  $\{v''_i\}_{i=1}^n$ . Moreover, vertices of  $K'' \setminus St(v'')$  are in the same  $v''$ -tier for any  $v'' \in E'_{i+1}$ . So we can repeat the previous process to deal with  $E'_{i+2}$ .

After finite many steps, we can assume for every point  $v' \in \{v'_i\}_{i=1}^n$ , vertices of  $K' \setminus St(v')$  are in the same  $v'$ -tier, thus  $K'$  satisfies the assumption of Lemma 5.7 and  $\pi \circ q_*$  induces a simplicial embedding  $s : F(\Gamma) \rightarrow F(\Gamma')$ . By considering the quasi-isometry inverse, we have a simplicial embedding  $s' : F(\Gamma') \rightarrow F(\Gamma)$ , thus  $F(\Gamma)$  and  $F(\Gamma')$  have the same number of vertices. Note that  $s(F(\Gamma))$  is a full subcomplex of  $F(\Gamma')$ , so  $s$  is actually a simplicial isomorphism and we have the following result:

**Theorem 5.9.** *If  $G(\Gamma)$  and  $G(\Gamma')$  are prime right-angled Artin groups, then they are quasi-isometric if and only if they are isomorphic.*

**5.2. Prime partition and sub-tiers.** Given right-angled Artin group  $G(\Gamma)$  of type II (not necessarily prime), our goal in the next two sections is to find a prime right-angled Artin group  $G(\Gamma')$  which is quasi-isometric to  $G(\Gamma)$ . Such  $G(\Gamma')$ , if exists, must be unique by Theorem 5.9.

Pick vertex  $\bar{v} \in F(\Gamma)$ , let  $\{\mathcal{C}_i\}_{i=1}^k$  be the collection of QII classes in  $F(\Gamma) \setminus St(\bar{v})$  and let  $(n_1, n_2, \dots, n_k)$  be the associated tuple. Let  $\{C_{ij}\}_{j=1}^{n_i}$  be the components in  $\mathcal{C}_i$  and let  $d = \gcd(n_1, n_2, \dots, n_k)$ . For each  $i$ , we choose a map  $f_i : \{C_{ij}\}_{j=1}^{n_i} \rightarrow \{1, 2, \dots, d\}$  such that for each  $1 \leq m \leq d$ , there are  $n_i/d$  elements in  $f_i^{-1}(m)$ . For  $1 \leq m \leq d$ , let  $\mathfrak{C}_m = \cup_{i=1}^k f_i^{-1}(m)$ . This partition of components of  $F(\Gamma) \setminus St(\bar{v})$  is called the *prime partition at  $\bar{v}$* . Each  $\mathfrak{C}_m$  is called a *prime factor at  $\bar{v}$* . The prime

partition comes together with an order, namely, we define  $\mathcal{C}_i \leq \mathcal{C}_j$  if  $i \leq j$ . Note that the prime partition is trivial if  $\bar{v}$  is prime. Now we fix a prime partition for every non-prime vertex in  $F(\Gamma)$ .

**Remark 5.10.** Let  $\alpha : F(\Gamma) \rightarrow F(\Gamma)$  be a simplicial automorphism. By consider the group automorphism of  $G(\Gamma)$  induced by  $\alpha$ , we deduce that the number of prime factors at  $\bar{v}$  and the number of prime factors at  $\alpha(\bar{v})$  are the same. However,  $\alpha$  may not map prime factors at  $\bar{v}$  to prime factors at  $\alpha(\bar{v})$ .

Let  $v \in \mathcal{P}(\Gamma)$  be a vertex such that  $\pi(v) = \bar{v}$  and let  $T$  be a  $v$ -tier. Recall that we have a map  $\Pi$  which maps  $v$ -branches to components of  $F(\Gamma) \setminus St(\bar{v})$ . This would give rise to a partition  $\{\Pi^{-1}(\mathfrak{C}_m) \cap T\}_{m=1}^d$  of  $v$ -branches in  $T$ . Each element in the partition is called a  $v$ -sub-tier.

**Remark 5.11.** Pick vertex  $x \in X(\Gamma)$  and let  $i_x : F(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  be the natural embedding. Then for vertices  $\bar{u}, \bar{v}, \bar{w} \in F(\Gamma)$ ,  $\bar{u}$  and  $\bar{v}$  are in different prime factors at  $\bar{w}$  if and only if  $i_x(\bar{u})$  and  $i_x(\bar{v})$  are in different  $i_x(\bar{w})$ -sub-tiers.

**Lemma 5.12.** *Let  $S_1$  and  $S_2$  be two  $v$ -sub-tiers. Then there exists a quasi-isometry  $q : X(\Gamma) \rightarrow X(\Gamma)$  such that the induces simplicial isomorphism  $q_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  satisfies*

- (1)  $q_*$  fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$ .
- (2)  $q_*(S_1) = S_2$  and  $q_*(S_2) = S_1$ .
- (3) For every  $v$ -branch  $B \subset S_1$ ,  $q_*(B)$  and  $B$  are QII.

*Proof.* To see this, note that there exist unique  $v$ -tiers  $T_1, T_2$  and  $1 \leq m_1, m_2 \leq d$  such that  $S_i = T_i \cap \Pi^{-1}(\mathfrak{C}_{m_i})$  for  $i = 1, 2$ . For each  $1 \leq i \leq k$ , pick a bijection between  $f_i^{-1}(m_1)$  and  $f_i^{-1}(m_2)$ , and this induces a bijection  $\bar{\Lambda}$  from components in  $\mathfrak{C}_{m_1}$  to components in  $\mathfrak{C}_{m_2}$ . By Corollary 3.14 (1),  $\bar{\Lambda}$  induces a bijection  $\Lambda$  from  $v$ -branches in  $S_1$  to  $v$ -branches in  $S_2$  such that  $B$  and  $\Lambda(B)$  are QII. We define  $q$  as follows. Set  $q(x) = x$  if  $x \in P_v$ . If  $x \notin P_v$ , let  $D$  be the component of  $X(\Gamma) \setminus P_v$  with  $x \in D$  and let  $B$  be the  $v$ -branch corresponding to  $D$  (see Corollary 3.26). If  $B$  is not inside  $S_1 \cup S_2$ , then set  $q(x) = x$ . Otherwise we assume  $B \subset S_1$ . Let  $B' = \Lambda(B)$  and let  $D'$  be the associated component of  $X(\Gamma) \setminus P_v$ . Let  $f$  be the elementary permutation (Definition 5.1) of  $B$  and  $B'$ . We can assume  $f(D) = D'$  (Corollary 3.26) and  $f$  is a  $(L, A)$ -quasi-isometry with  $L$  and  $A$  independent of  $B \subset S_1$  (see the discussion after Lemma 5.2). Set  $q(x) = f(x)$  in this case. Then  $q$  is a quasi-isometry and satisfies all the requirements.  $\square$

**Remark 5.13.** Lemma 5.6 is still true if we replaced  $v_i$ -tier by  $v_i$ -sub-tier in (3), the same proof goes through.

**Lemma 5.14.** *If  $G(\Gamma)$  is of type II, then given any two vertices  $v_1, v_2 \in \mathcal{P}(\Gamma)$ , there only exists finite many vertices  $w$  such that  $v_1|_w v_2$ .*

*Proof.* We need an auxiliary result: pick vertex  $x \in X(\Gamma)$  and  $v \in \mathcal{P}(\Gamma) \setminus (F(\Gamma))_x$ , let  $\bar{w} \in \Gamma$  and let  $S_{\bar{w}} \subset (F(\Gamma))_x$  be the lift of  $St(\bar{w}) \subset F(\Gamma)$ . Then  $S_{\bar{w}} \setminus St(v) \neq \emptyset$ . Suppose the contrary is true, put  $\bar{v} = \pi(v)$ , then  $St(\bar{w}) \subset St(\bar{v})$ , hence  $\bar{v} \in St(\bar{w})$ . Let  $v' \in (F(\Gamma))_x$  be the lift of  $\bar{v}$ . Then  $v' \in S_{\bar{w}} \subset St(v)$ . Note that  $d(v', v) = 1$  is impossible since  $\pi(v') = \pi(v)$ , so  $v' = v$ , which is contradictory to  $v \notin (F(\Gamma))_x$ .

Pick an edge path  $\omega$  which connects a vertex in  $P_{v_1}$  to a vertex in  $P_{v_2}$ . Let  $\{x_i\}_{i=0}^n$  be consecutive vertices in  $\omega$ , and let  $l_i$  be the standard geodesic containing  $x_{i-1}$  and  $x_i$ . Let  $K_i = (F(\Gamma))_{x_i}$  and  $K = \cup_{i=0}^n K_i$ . Then  $v_1 \in K_0$  and  $v_2 \in K_n$ .

It suffices to show that for any vertex  $v \notin K$ ,  $v_1$  and  $v_2$  are in the same  $v$ -branch. To see this, note that  $\pi(K_{i-1} \cap K_i) = St(\pi(\Delta(l_i)))$ , so for  $1 \leq i \leq n$ , there exists vertex  $w_i$  such that  $w_i \in (K_{i-1} \cap K_i) \setminus St(v)$  by the auxiliary result above. By remark 3.8,  $w_i w_{i+1}|_v$  for  $1 \leq i \leq n-1$ ,  $v_1 w_1|_v$  and  $w_n v_2|_v$ , so  $v_1 v_2|_v$ .  $\square$

The reader can proceed directly to Section 5.3 and come back to the following technical lemma later.

Let  $\bar{v} \in F(\Gamma)$  be a non-prime vertex and let  $K \subset F(\Gamma)$  be a subcomplex containing  $\bar{v}$  such that for any vertex  $\bar{u} \in F(\Gamma) \setminus lk(\bar{v})$ ,  $K \setminus St(\bar{u})$  is inside a prime factor at  $\bar{u}$ . Suppose in addition that  $K \setminus St(\bar{v}) \neq \emptyset$  and let  $\mathfrak{C}_1$  be the prime factor at  $\bar{v}$  that contains  $K \setminus St(\bar{v})$ . Let  $\mathfrak{C}_2$  be a different prime factor at  $\bar{v}$ .

Pick vertex  $x \in X(\Gamma)$  and let  $K', v, \mathfrak{C}'_1$  and  $\mathfrak{C}'_2$  be the lift of  $K, \bar{v}, \mathfrak{C}_1$  and  $\mathfrak{C}_2$  in  $(F(\Gamma))_x \subset \mathcal{P}(\Gamma)$  respectively. For  $i = 1, 2$ , let  $S_i$  be the  $v$ -sub-tier that contains  $\mathfrak{C}'_i$ . Let  $q : X(\Gamma) \rightarrow X(\Gamma)$  be a quasi-isometry such that  $q_*$  permutes  $S_1$  and  $S_2$  and fixes every points in  $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$  (Lemma 5.12).

**Lemma 5.15.** *There exists a quasi-isometry  $h : X(\Gamma) \rightarrow X(\Gamma)$  such that*

- (1)  $h_*$  permutes  $S_1$  and  $S_2$  and fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$ .
- (2) The projection map  $\pi : \mathcal{P}(\Gamma) \rightarrow F(\Gamma)$  restricted on  $K' \cup h_*(K')$  is injective.
- (3) Let  $M = \pi(K' \cup h_*(K'))$ . Let  $\bar{h}_* : K \rightarrow \pi(h_*(K'))$  be the simplicial isomorphism induced by  $h_*$ . Pick vertex  $\bar{u} \in F(\Gamma)$ . Then
  - (a) If  $d(\bar{u}, \bar{v}) \geq 2$ , then  $M \setminus St(\bar{u})$  is contained in one prime factor at  $\bar{u}$ .
  - (b) If  $d(\bar{u}, \bar{v}) = 1$ , then  $\bar{r} \in K \setminus St(\bar{u})$  if and only if  $\bar{h}_*(\bar{r}) \in \bar{h}_*(K) \setminus St(\bar{u})$ . In this case,  $\bar{r}$  and  $\bar{h}_*(\bar{r})$  are in the same prime factor at  $\bar{u}$ .
  - (c) If  $K$  is a full subcomplex of  $F(\Gamma)$ , then  $\bar{h}_*(K)$  is also a full subcomplex.

*Proof.* We assume  $K$  is a full subcomplex. The general case follows from this special case by considering the full subcomplex spanned by  $K$ .

Let  $L = K' \cup q_*(K')$ . By Lemma 5.14, there are only finitely many vertices  $w \in \mathcal{P}(\Gamma)$  such that  $St(w)$  separates two vertices in  $L$ . Denote these vertices by  $\{w_i\}_{i=1}^n$ . We claim if  $St(w)$  separates two vertices in  $K'$  and  $d(w, v) \geq 2$ , then vertices of  $q'_*(K') \setminus St(w)$  are in the same  $w$ -branch. To see this, suppose  $v_1|_w v_2$  for  $v_1, v_2 \in K'$ , then either  $v_1|_w v$  or  $v_2|_w v$ , so  $v_1 w|_v$  or  $v_2 w|_v$  by Lemma 5.8. Then the claim follows from Lemma 5.5. Note that the claim is also true if we switch the role of  $K'$  and  $q'_*(K')$ .

By the above claim, we can divide  $\{w_i\}_{i=1}^n$  into the following 4 groups.

- (1)  $w_1 = v$ .
- (2)  $w_i \in lk(v)$  if and only if  $2 \leq i \leq n_1$ .
- (3)  $St(w_i)$  separates  $v$  from some vertex in  $K'$  if and only if  $n_1 + 1 \leq i \leq n_2$ .
- (4)  $St(w_i)$  separates  $v$  from some vertex in  $q_*(K')$  if and only if  $n_2 + 1 \leq i \leq n$ .

Note that  $q_*$  induces a bijection between  $\{w_i\}_{i=n_1+1}^{n_2}$  and  $\{w_i\}_{i=n_2+1}^n$ . Let  $k = n_2 - n_1 = n - n_2$ . We also assume  $q_*(w_i) = w_{i+k}$  for  $n_1 + 1 \leq i \leq n_2$ .

Let  $D = \{w_i\}_{i=1}^{n_2}$ . Then  $D \setminus St(w_i)$  stays inside a  $w_i$ -branch for  $i > n_2$ . To see this, let  $B$  be the  $w_i$ -branch that contains  $v$  and let  $w_{i_0} \in D \setminus St(w_i)$ . It is clear that  $w_{i_0} \in B$  if  $i_0 \leq n_1$ . If  $n_1 < i_0 \leq n_2$ , by above discussion, there exists  $u \in K'$  such that  $w_{i_0} u|_v$ , similarly, there exists  $u' \in q_*(K')$  such that  $w_i u'|_v$ . But  $u|_v u'$ , so  $w_{i_0}|_v w_i$ , and by Lemma 5.8, we have  $w_{i_0} v|_{w_i}$  and  $w_{i_0} \in B$ . This discussion also implies  $\{w_i\}_{i=n_1+1}^{n_2} \subset S_1$  and  $\{w_i\}_{i=n_2+1}^n \subset S_2$ .

Let  $\{B_\lambda\}_{\lambda \in \Lambda}$  be the collection of  $w_i$ -branches that contain vertices of  $K'$ , here  $i$  ranges over all values between 2 and  $n_1$ . By Lemma 5.6, we can post-compose

$q_*$  with a simplicial isomorphism  $f_*$  to obtain a map  $q'_* = f_* \circ q_*$  that satisfies the conclusions of Lemma 5.6. By Remark 5.13, we can assume that for vertex  $u \in K' \setminus St(w_i)$  ( $2 \leq i \leq n_1$ ),  $u$  and  $q'_*(u)$  are in the same  $w_i$ -sub-tier. Let  $L' = K' \cup q'_*(K')$ . Note that if  $B_\lambda \subset \mathcal{P}(\Gamma) \setminus St(v)$ , then  $B_\lambda \subset S_1$ , so by the construction in Lemma 5.6,  $f_*$  is a composition of elementary permutations that happen inside  $S_2$ , hence  $f_*$  fixes every point in  $S_1$ , in particular,  $f_*(K') = K'$  and  $f_*(L) = L'$ . Let  $w'_i = f_*(w_i)$ . Then  $\{w'_i\}_{i=1}^n$  is the collection of vertices such that  $St(w)$  separates two vertices in  $L'$ . We divide  $\{w'_i\}_{i=1}^n$  into 4 groups as before and this partition coincides with the partition induced by  $f_*$ . Since  $w_i \in S_1$  for  $n_1 + 1 \leq i \leq n_2$ , so  $w'_i = w_i$  for  $i \leq n_2$ . Moreover, the property in the previous paragraph also holds for  $\{w'_i\}_{i=1}^n$ .

We claim for  $n_1 < i \leq n_2$ , vertices of  $L' \setminus St(w'_i)$  are in the same  $w'_i$ -sub-tier. Actually, by Remark 3.8,  $w'_i \in (F(\Gamma))_x$  for  $n_1 < i \leq n_2$ . Recall that  $K \setminus St(\pi(w'_i))$  is inside a prime factor at  $\pi(w'_i)$ , so vertices of  $K' \setminus St(w'_i)$  are inside a  $w'_i$ -sub-tier.  $S_2$  contains vertices of  $q'_*(K') \setminus St(v)$ , but  $w'_i \in S_1$ , so vertices of  $q'_*(K') \setminus St(v)$  and  $v$  are in the same  $w'_i$ -branch by Lemma 5.5, thus the claim follows.

Let  $E_0 = \{w'_i\}_{i=1}^{n_2}$  and  $E = \{w'_i\}_1^n$ . Then  $E_0$  is tight in  $E$  by previous discussion. Let  $E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \dots \subsetneq E_k = E$  be a tight filtration. Up to reordering, we assume  $w'_{i+n_2} = E_i \setminus E_{i-1}$  for  $1 \leq i \leq k$  and  $q'_*(w'_i) = w'_{i+k}$  for  $n_1 < i \leq n_2$ . Suppose there is an integer  $m$  ( $0 \leq m \leq k$ ) such that

- (1)  $q'_*$  permutes  $S_1$  and  $S_2$  and fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$ .
- (2) For  $2 \leq i \leq n_1$  and vertex  $u \in K' \setminus St(w'_i)$ ,  $u$  and  $q'_*(u)$  are in the same  $w'_i$ -sub-tier.
- (3) For  $n_1 < i \leq n_2 + m$ , vertices of  $L' \setminus St(w'_i)$  are contained in one  $w'_i$ -sub-tier.

Such  $m$  always exists, since (1) and (2) is always true and (3) is true when  $m = 0$ . Our goal to is modify the map  $q'_*$  such that  $m = k$ . Now we assume  $m < k$  and argue by induction.

Let  $a = n_1 + m + 1$  and  $b = n_2 + m + 1$ . Since vertices of  $K' \setminus St(w'_a)$  stay inside a  $w'_a$ -sub-tier and  $w'_b = q'_*(w'_a)$ , there is a simplicial isomorphism  $g_*$  which is a composition of finitely many elementary permutations of  $w'_b$ -branches such that (1) vertices in  $g_*(q'_*(K')) \setminus St(w'_b)$  are in the same  $w'_b$ -sub-tier; (2) let  $B'$  be the  $w'_b$ -branch that contains  $v$ , then  $g_*$  fixes every point in  $B'$ . Lemma 5.5 implies vertices of  $K' \setminus St(w'_b)$  are in  $B'$ , so  $g_*$  fixes every point in  $K'$ . Moreover, the tightness of  $E_m$  implies  $g_*(w'_i) = w'_i$  for  $1 \leq i \leq b$ . By Lemma 5.6 and Remark 5.13, we can assume in addition that (3) for any vertex  $t \in St(w'_b) \cap E_m$  and any  $t$ -branch  $A$  with  $A \cap L' \neq \emptyset$ ,  $g_*(A)$  and  $A$  are in the same  $t$ -sub-tier.

Let  $L'' = K' \cup g_*(q'_*(K'))$ . Then  $g_*(L') = L''$ . Let  $w''_i = g_*(w'_i)$ . Then  $\{w''_i\}_{i=1}^n$  is the collection of vertices such that  $St(w''_i)$  separates two vertices in  $L''$ . Moreover,  $g_*(E_0) \subsetneq g_*(E_1) \subsetneq \dots \subsetneq g_*(E_k) = g_*(E)$  is a tight filtration. Note that  $g_*(E_i) = E_i$  for  $i \leq m + 1$ . We claim

- (1)  $g_* \circ q'_*$  permutes  $S_1$  and  $S_2$  and fixes every vertex in  $\mathcal{P}(\Gamma) \setminus (S_1 \cup S_2)$ .
- (2) For  $2 \leq i \leq n_1$  and vertex  $u \in K' \setminus St(w''_i)$ ,  $u$  and  $g_*(q'_*(u))$  are in the same  $w''_i$ -sub-tier.
- (3) For  $n_1 < i \leq b$ , vertices of  $L'' \setminus St(w''_i)$  are contained in one  $w''_i$ -sub-tier.

(1) follows from property (2) of  $g_*$  and Lemma 5.5. Now we assume  $2 \leq i \leq n_1$ , then  $w''_i = w'_i$ . If  $d(w'_i, w'_b) \geq 2$ , since  $g_*$  fixes every point in  $B'$ , we can show  $g_*$  induces trivial permutation of  $w'_i$ -branches as in the proof of Theorem 5.9, then (2) follows from the induction assumption. If  $d(w'_i, w'_b) = 1$ , since  $q'_*(u) \in L'$ ,  $q'_*(u)$

and  $g_*(q'_*(u))$  are in the same  $w'_i$ -sub-tier by property (3) of  $g_*$ . But  $u$  and  $q'_*(u)$  are in the same  $w'_i$ -sub-tier by induction, thus (2) follows.

It remains to verify (3). Suppose  $n_1 < i \leq b$ , we still have  $w''_i = w'_i$  and  $w''_b = w'_b$ . Since  $g_*(L') = L''$ , the case  $i < b$  and  $d(w'_i, w'_b) = 1$  follows from induction assumption and property (3) of  $g_*$ . If  $i < b$  and  $d(w'_i, w'_b) = 2$ , then  $g_*$  induces trivial permutation of  $w'_i$ -branches and (3) follows from the induction assumption. If  $i = b$ , by Lemma 5.5, vertices of  $K' \setminus St(w'_b)$  and  $v$  are in the same  $w'_b$ -branch, then (3) follows from property (1) of  $g_*$ .

After applying the above induction process for finite many times, we obtain a simplicial isomorphism  $h_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma)$  which satisfies (1) in Lemma 5.15. Moreover, let  $\tilde{L} = K' \cup h_*(K')$  and let  $\{\tilde{w}_i\}_{i=1}^n$  be the collection of vertices such that  $St(\tilde{w}_i)$  separates two vertices of  $\tilde{L}$ . Then

- (1) For  $2 \leq i \leq n_1$  and vertex  $u \in K' \setminus St(\tilde{w}_i)$ ,  $u$  and  $h_*(u)$  are in the same  $\tilde{w}_i$ -sub-tier.
- (2) For  $n_1 < i \leq n$ , vertices of  $\tilde{L} \setminus St(\tilde{w}_i)$  are contained in one  $\tilde{w}_i$ -sub-tier.

Let  $2 \leq i \leq n_1$ . Since  $h_*(\tilde{w}_i) = \tilde{w}_i$  and vertices of  $K' \setminus St(\tilde{w}_i)$  are contained in one  $\tilde{w}_i$ -tier, (1) implies actually vertices of  $\tilde{L} \setminus St(\tilde{w}_i)$  are contained in one  $\tilde{w}_i$ -tier. Thus  $\tilde{L}$  satisfies the assumption of Lemma 5.7 and (2) of Lemma 5.15 follows. Note that  $K'$  is a full subcomplex, so is  $h_*(K')$ , then  $\tilde{h}_*(K) = \pi(h_*(K'))$  is a full subcomplex.

Pick vertex  $x_0 \in \cap_{w \in \tilde{L}} P_w$  ( $x_0$  may not equal to  $x$ ), then  $\tilde{L} \subset (F(\Gamma))_{x_0}$ . Let  $i_{x_0} : F(\Gamma) \rightarrow (F(\Gamma))_{x_0} \subset \mathcal{P}(\Gamma)$  be the natural embedding. Then  $i_{x_0}(M) = \tilde{L}$  and (3a) follows from property (2) of  $h_*$  and Remark 5.11. Now we look at (3b). For vertex  $\bar{k} \in K$ ,

$$\begin{aligned} d(\bar{k}, \bar{u}) &= d(i_{x_0}(\bar{k}), i_{x_0}(\bar{u})) = d(h_* \circ i_{x_0}(\bar{k}), i_{x_0}(\bar{u})) \\ &= d(\pi \circ h_* \circ i_{x_0}(\bar{k}), \pi \circ i_{x_0}(\bar{u})) = d(\tilde{h}_*(\bar{k}), \bar{u}). \end{aligned}$$

The first and the third equality follow from Lemma 2.13, and the second equality holds since  $h_*$  fixes  $i_{x_0}(\bar{u})$ . Thus the first part of (3b) is true. The rest of (3b) follows from property (1) of  $h_*$ .  $\square$

**5.3. An wall space.** In this section, we will construct a prime right-angled Artin group which is a special subgroup of  $G(\Gamma)$ .

**Step 1:** We show the prime partition induces a wall-space structure on  $F(\Gamma)$  and we construct the associated  $CAT(0)$  cube complex.

Pick non-prime vertex  $\bar{v} \in F(\Gamma)$  and let  $\{\mathfrak{C}_j\}_{j=1}^d$  be the prime factors at  $\bar{v}$ . A  $\bar{v}$ -halfspace of  $F(\Gamma)$  is a full subcomplex of form  $St(\bar{v}) \cup (\cup_{j=1}^m \mathfrak{C}_j)$  or  $St(\bar{v}) \cup (\cup_{j=m+1}^d \mathfrak{C}_j)$  with  $1 \leq m < d$ . Let  $H = St(\bar{v}) \cup (\cup_{j=1}^m \mathfrak{C}_j)$  (or  $St(\bar{v}) \cup (\cup_{j=m+1}^d \mathfrak{C}_j)$ ). We define the complement of  $H$ , denoted by  $H^c$ , to be  $St(\bar{v}) \cup (\cup_{j=m+1}^d \mathfrak{C}_j)$  (or  $St(\bar{v}) \cup (\cup_{j=1}^m \mathfrak{C}_j)$ ). A  $\bar{v}$ -wall of  $F(\Gamma)$  is a pair of halfspaces  $(H, H^c)$ . Let  $\mathcal{H}(\Gamma)$  be the collection of pairs  $(\bar{v}, H)$  such that  $\bar{v}$  is non-prime and  $H$  is a  $\bar{v}$ -halfspace. If there is another pair  $(\bar{v}', H')$  in  $\mathcal{H}(\Gamma)$  such that  $H = H'$  and  $\bar{v} \neq \bar{v}'$ , then  $(\bar{v}', H')$  and  $(\bar{v}, H)$  are viewed as different elements in  $\mathcal{H}(\Gamma)$ . Let  $\mathcal{W}(\Gamma)$  be the collection of triples  $(\bar{v}, H, H^c)$  such that  $(H, H^c)$  is a  $\bar{v}$ -wall. Occasionally, we will omit  $\bar{v}$  when there is no ambiguity.

Define two halfspaces  $(\bar{v}_1, H_1), (\bar{v}_2, H_2) \in \mathcal{H}(\Gamma)$  are compatible if  $d(\bar{v}_1, \bar{v}_2) = 1$  or  $(H_1 \cap H_2) \not\subseteq St(\bar{v}_1)$ . Note that if  $d(\bar{v}_1, \bar{v}_2) \geq 2$ , then  $(H_1 \cap H_2) \not\subseteq St(\bar{v}_1)$  implies  $(H_1 \cap H_2) \not\subseteq St(\bar{v}_2)$  and vice versa, and in this case exactly one of the following three possibilities is true: (1)  $\bar{v}_1 \in H_2$  and  $\bar{v}_2 \in H_1$ ; (2)  $H_2 \subsetneq H_1$ ; (3)  $H_1 \subsetneq H_2$ .

To see this, let  $C_1$  (or  $C_2$ ) be the component of  $F(\Gamma) \setminus St(\bar{v}_1)$  (or  $F(\Gamma) \setminus St(\bar{v}_2)$ ) that contains  $\bar{v}_2$  (or  $\bar{v}_1$ ). If  $C_1 \cap H_1 = C_2 \cap H_2 = \emptyset$ , then  $H_2 \subset C_1 \cup St(\bar{v}_2)$  by Lemma 5.5, hence  $H_1 \cap H_2 \subset H_1 \cap (C_1 \cup St(\bar{v}_2)) = H_1 \cap St(\bar{v}_2) = St(\bar{v}_1) \cap St(\bar{v}_2) \subset St(\bar{v}_1)$ , which yields a contradiction. If  $C_i \cap H_i \neq \emptyset$  for  $i = 1, 2$ , then actually  $C_i \subset H_i$  and case (1) holds. Moreover, neither  $H_2 \subset H_1$  nor  $H_1 \subset H_2$  is true in this case by Lemma 5.5. If  $C_1 \subset H_1$  and  $C_2 \cap H_2 = \emptyset$ , then  $H_2 \subset C_1 \cup St(\bar{v}_2) \subset H_1$ . Note that  $\bar{v}_1 \in H_1 \setminus H_2$ , hence case (2) is true. Similarly,  $C_2 \subset H_2$  and  $C_1 \cap H_1 = \emptyset$  implies case (3). It easy to see  $(H_1 \cap H_2) \not\subseteq St(\bar{v}_2)$  holds in each case. We also deduce that if  $d(\bar{v}_1, \bar{v}_2) \geq 2$ , then  $H_1$  and  $H_2$  are compatible if and only if  $C_1 \cap H_1 = C_2 \cap H_2 = \emptyset$  is not true, hence (under the assumption  $d(\bar{v}_1, \bar{v}_2) \geq 2$ ) (1) is true if and only if  $C_i \subset H_i$  for  $i = 1, 2$ ; (2) is true if and only if  $C_1 \subset H_1$  and  $C_2 \cap H_2 = \emptyset$  and (3) is true if and only if  $C_2 \subset H_2$  and  $C_1 \cap H_1 = \emptyset$ .

We define  $(\bar{v}_1, H_1) \leq (\bar{v}_2, H_2)$  if  $d(\bar{v}_1, \bar{v}_2) \neq 1$  and  $H_1 \subset H_2$ . It follows from the above discussion that  $\leq$  is antisymmetric. Pick  $(\bar{v}_3, H_3) \in \mathcal{H}(\Gamma)$  such that  $(\bar{v}_2, H_2) \leq (\bar{v}_3, H_3)$ , if two of  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  are the same, then  $(\bar{v}_1, H_1) \leq (\bar{v}_3, H_3)$  by definition. If  $\bar{v}_1, \bar{v}_2, \bar{v}_3$  are pairwise distinct, let  $C_1$  and  $C_2$  be as above and let  $C'_2$  be the component of  $F(\Gamma) \setminus St(\bar{v}_2)$  that contains  $\bar{v}_3$ . Since  $H_1 \subsetneq H_2$  and  $H_2 \subsetneq H_3$ , then  $C_1 \cap H_1 = \emptyset$ ,  $C_2 \subset H_2$  and  $C'_2 \cap H_2 = \emptyset$  by above discussion. Thus  $\bar{v}_1$  and  $\bar{v}_3$  are in different component of  $F(\Gamma) \setminus St(\bar{v}_2)$  and  $d(\bar{v}_1, \bar{v}_3) \geq 2$ , which implies  $(\bar{v}_1, H_1) \leq (\bar{v}_3, H_3)$ . It follows that  $\leq$  is a partial order.

Note that  $(\bar{v}_1, H_1) \leq (\bar{v}_2, H_2)$  implies  $(\bar{v}_2, H_2^c) \leq (\bar{v}_1, H_1^c)$ , hence  $\mathcal{H}(\Gamma)$  is a pocset. The case  $\bar{v}_1 = \bar{v}_2$  is clear. If  $d(\bar{v}_1, \bar{v}_2) \geq 2$ , then  $H_1 \cap C_1 = \emptyset$  and  $C_2 \subset H_2$ , hence  $C_1 \subset H_1^c$  and  $C_2 \cap H_2^c = \emptyset$ , which implies  $(\bar{v}_2, H_2^c) \leq (\bar{v}_1, H_1^c)$ .

We claim the following are equivalent.

- (1)  $(\bar{v}_1, H_1)$  and  $(\bar{v}_2, H_2)$  are not compatible.
- (2)  $d(\bar{v}_1, \bar{v}_2) \neq 1$  and  $(\bar{v}_1, H_1) \leq (\bar{v}_2, H_2^c)$ .
- (3)  $d(\bar{v}_1, \bar{v}_2) \neq 1$  and  $(\bar{v}_2, H_2) \leq (\bar{v}_1, H_1^c)$ .

It follows from this claim that  $U \subset \mathcal{H}(\Gamma)$  is an ultrafilter if and only if (1) for each pair  $(\bar{v}, H)$  and  $(\bar{v}, H^c)$ ,  $U$  contains exactly one of them; (2) every pair of halfspaces in  $U$  is compatible. To see the claim, let us assume  $d(\bar{v}_1, \bar{v}_2) \geq 2$ . Then  $(\bar{v}_1, H_1)$  and  $(\bar{v}_2, H_2)$  are not compatible  $\Leftrightarrow C_1 \cap H_1 = C_2 \cap H_2 = \emptyset \Leftrightarrow C_1 \cap H_1 = \emptyset$  and  $C_2 \subset H_2^c \Leftrightarrow (\bar{v}_1, H_1) \leq (\bar{v}_2, H_2^c)$ .

**Lemma 5.16.** *For arbitrary simplex  $g \subset F(\Gamma)$ , there exists an ultrafilter  $U$  such that the intersection of halfspaces in  $U$  contains  $g$ .*

*Proof.* Let  $E'$  be the collection of non-prime vertices in  $F(\Gamma)$  and let  $G$  be the collection of vertices in  $g$ . Let  $E = E' \cup G$  and let  $E_1 \subsetneq E_2 \subsetneq \dots \subsetneq E_n = E$  be a tight filtration of  $E$  (see the discussion after Lemma 5.8). Denote  $\bar{u}_i = E_i \setminus E_{i-1}$ . We can assume in addition that  $\bar{u}_i \in G$  if and only if  $i \leq n_1$  and  $\bar{u}_i \in E'$  if and only if  $i \geq n_2$ . For  $i \geq n_2$ , if  $E_{i-1} \setminus St(\bar{u}_i) \neq \emptyset$ , let  $C_i$  be the component of  $F(\Gamma) \setminus St(\bar{u}_i)$  that contains  $E_{i-1} \setminus St(\bar{u}_i)$  (this is possible by our choice of  $E_i$ ). If  $E_{i-1} \setminus St(\bar{u}_i) = \emptyset$ , let  $C_i$  be an arbitrary component. We define  $U$  by choosing the unique halfspace that contains  $C_i$  in each  $\bar{u}_i$ -wall for  $i \geq n_2$ . It clear that the intersection of halfspaces in  $U$  contains  $g$ . It remains to show two halfspaces  $(\bar{u}_i, H_1), (\bar{u}_j, H_2) \in U$  are compatible. The case  $d(\bar{v}_1, \bar{v}_2) \leq 1$  is trivial. We assume  $d(\bar{u}_i, \bar{u}_j) \geq 2$ . Suppose  $i < j$ , then  $\bar{u}_i \subset C_j \subset H_2$ , hence  $\bar{u}_i \in (H_1 \cap H_2) \setminus St(\bar{u}_j)$ . It follows that  $U$  is an ultrafilter.  $\square$

Let  $U = \{(\bar{u}_\lambda, H_\lambda)\}_{\lambda \in \Lambda}$  be an ultrafilter and let  $A = \cap_{\lambda \in \Lambda} H_\lambda$ . We claim  $A \neq \emptyset$ . Note that if  $(\bar{u}_\lambda, H_\lambda)$  is minimal in  $U$ , then  $\bar{u}_\lambda \in A$ . Suppose the contrary is true, then there exists  $(\bar{u}_{\lambda'}, H_{\lambda'}) \in U$  such that  $\bar{u}_\lambda \notin H_{\lambda'}$ , in particular  $d(\bar{u}_{\lambda'}, \bar{u}_\lambda) \geq 2$ . By compatibility of  $H_\lambda$  and  $H_{\lambda'}$ , we must have  $H_{\lambda'} \subsetneq H_\lambda$ , which contradicts the minimality of  $(\bar{u}_\lambda, H_\lambda)$ .

Let  $X$  be the  $CAT(0)$  cube complex obtained from the pocset  $\mathcal{H}(\Gamma)$  as in Theorem 2.9. Let  $\Phi$  be the pocset isomorphism from the collection of halfspaces in  $X$  to  $\mathcal{H}(\Gamma)$  as in Theorem 2.9. Then  $\Phi$  induces a bijective map from hyperplanes of  $X$  to  $\mathcal{W}(\Gamma)$ , which is also denoted by  $\Phi$ . Let  $\{x_i\}_{i=1}^r$  be the collection of vertices in  $X$ , and let  $\{U(x_i)\}_{i=1}^r$  be the corresponding ultrafilters. Let  $\Phi(x_i)$  be the intersection of halfspaces in  $U(x_i)$ . Then

- (1) For any vertex  $\bar{u} \in F(\Gamma)$ ,  $\Phi(x_i) \setminus St(\bar{u})$  is contained in a prime factor at  $\bar{u}$ .
- (2)  $\cup_{i=1}^r \Phi(x_i) = F(\Gamma)$ .
- (3)  $\Phi(x_i) \neq \emptyset$  for all  $i$ .

(2) follows from Lemma 5.16 and (3) follows from the above discussion.

Recall that two distinct walls  $(\bar{v}_1, H_1, H_1^c), (\bar{v}_2, H_2, H_2^c) \in \mathcal{W}(\Gamma)$  are *transverse* if none of  $(\bar{v}_1, H_1) < (\bar{v}_2, H_2)$ ,  $(\bar{v}_1, H_1) < (\bar{v}_2, H_2^c)$ ,  $(\bar{v}_2, H_2) < (\bar{v}_1, H_1)$  and  $(\bar{v}_2, H_2) < (\bar{v}_1, H_1^c)$  is true. Thus two such walls are transverse if and only if  $d(\bar{v}_1, \bar{v}_2) = 1$  (note that when  $d(\bar{v}_1, \bar{v}_2) = 1$ , even if  $H_1 \subset H_2$ , we still have  $(\bar{v}_1, H_1) \not\leq (\bar{v}_2, H_2)$  by our definition). It follows that if  $h'_1$  and  $h'_2$  is a pair of crossing hyperplanes in  $X$  and  $\Phi(h'_i)$  is a  $\bar{v}_i$ -wall for  $i = 1, 2$ , then  $d(\bar{v}_1, \bar{v}_2) = 1$ .

For each subcomplex  $A \subset X$ , we define  $\Phi(A) = \cup_{x \in A} \Phi(x)$  ( $x$  is a vertex). If  $A$  is convex, then  $\Phi(A)$  is a full subcomplex. This follows from the following observation. Let  $\{h_i\}_{i=1}^t$  be the collection of halfspaces in  $X$  with  $A \subset h_i$  and let  $\Phi(h_i) = (\bar{w}_i, h'_i)$ . Suppose  $K = \cap_{i=1}^t h'_i$ . Since each  $h'_i$  is a full subcomplex, so is  $K$ . It suffices to show  $\Phi(A) = K$ .  $\Phi(A) \subset K$  is clear. Let  $\mathcal{W}'(\Gamma)$  be the  $\Phi$ -image of hyperplanes in  $X$  that intersect  $A$  and let  $\mathcal{H}'(\Gamma)$  be the corresponding collection of halfspaces. Then  $\mathcal{H}'(\Gamma)$  is a sub-pocset of  $\mathcal{H}(\Gamma)$ . We claim  $U' \subset \mathcal{H}'(\Gamma)$  is an ultrafilter of  $\mathcal{H}'(\Gamma)$  if and only if  $U' \cup \{h'_i\}_{i=1}^t$  is an ultrafilter of  $\mathcal{H}(\Gamma)$ . To see this, we can use the pocset isomorphism  $\Phi$  between the halfspaces of  $X$  and  $\mathcal{H}(\Gamma)$  to translate this statement to a statement about halfspaces of  $X$ , which becomes obvious. We also deduce that  $U' \cup \{h'_i\}_{i=1}^t$  corresponds to a vertex in  $A$ . Thus there is an isometric embedding from the  $CAT(0)$  cube complex associated with  $\mathcal{H}'(\Gamma)$  to  $X$ , whose image is exactly  $A$ . Let  $\{U'_i\}_{i=1}^l$  be the collection of ultrafilters on  $\mathcal{H}'(\Gamma)$  and let  $K_i$  be the intersection of halfspaces in  $U'_i$ . Then we can prove  $\cup_{i=1}^l K_i = F(\Gamma)$  as in Lemma 5.16. It follows that  $K = K \cap (\cup_{i=1}^l K_i) = \cup_{i=1}^l (K \cap K_i)$ , but  $K \cap K_i = U(x)$  for some vertex  $x \in A$ , so  $K \subset \Phi(A)$ .

**Step 2:** We study the relation between  $\Phi(x)$  and  $F(\Gamma)$  ( $x \in F(\Gamma)$  is a vertex). For this purpose, we define a filtration for  $X$  as well as for  $F(\Gamma)$ . Such filtration is motivated by the generalized star extension introduced in [Hua14a, Section 6.3].

We define a chain of convex subcomplexes in  $X$  by induction. Pick a vertex  $x \in X$  and set  $L_1 = \{x\}$ . Suppose we have already defined  $L_i$ . If  $L_i = X$ , then we stop; if  $L_i \subsetneq X$ , pick an edge  $e_i$  such that  $e_i \cap L_i$  is a vertex and let  $L_{i+1}$  be the convex hull of  $L_i \cup e_i$ . Let  $\{L_i\}_{i=1}^s$  be the resulting collection of convex subcomplexes. Here is an alternative way of describing  $L_{i+1}$ . Suppose  $h_i$  is the hyperplane dual to  $e_i$  and  $N_i$  is the carrier of  $h_i$ . Then  $h_i \cap L_i = \emptyset$  by the convexity of  $L_i$ . Let  $M_i$  be the copy of  $(L_i \cap N_i) \times [0, 1]$  inside  $N_i$ . Then  $L_{i+1} = L_i \cup M_i$ .

Now we look at the relation between  $\Phi(L_i)$  and  $\Phi(L_{i+1})$ . For  $j = 1, 2$ , let  $M_{ij}$  be the subcomplex of  $M_i$  of form  $(L_i \cap N_i) \times \{j - 1\}$ . We assume  $M_{i1} = L_i \cap N_i$  and let  $p : M_{i1} \rightarrow M_{i2}$  be the map induced by parallelism. Suppose  $(\bar{v}, H_i) \in \mathcal{H}(\Gamma)$  is the element corresponding to the halfspace of  $h_i$  that contains  $L_i$ . Then  $\Phi(M_{i1}) \subset \Phi(K) \subset H_i$  and  $\Phi(M_{i2}) \subset H_i^c$ . For any vertex  $x \in M_{i1}$ ,  $(\bar{v}, H_i)$  is a minimal element in  $U(x)$ , so  $\bar{v} \in \Phi(x) \subset \Phi(M_{i1})$ . Similarly,  $\bar{v} \in \Phi(M_{i2})$ .

We claim there exist  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  which are prime factors at  $\bar{v}$  such that  $\mathfrak{C}_1 \subset H_i$ ,  $\mathfrak{C}_2 \subset H_i^c$  and  $\Phi(M_{ij}) \setminus St(\bar{v}) \subset \mathfrak{C}_j$  for  $j = 1, 2$ . Pick adjacent vertices  $x_1, x_2 \in M_{i1}$ , then there exists  $(\bar{v}, H'_i) \in U(x_1)$  such that  $(H_i \cap H'_i) \setminus St(\bar{v})$  is a prime factor at  $\bar{v}$ . Denote this prime factor by  $\mathfrak{C}_1$ , then  $\Phi(x_1) \setminus St(\bar{v}) \subset \mathfrak{C}_1$ . Let  $h$  be the hyperplane dual to the edge joining  $x_1$  and  $x_2$  and let  $\Phi(h) = (\bar{w}, H, H^c)$ . Then  $U(x_2) = (U(x_1) \setminus \{H\}) \cup \{H^c\}$ . Since  $h$  crosses  $h_i$ ,  $d(\bar{w}, \bar{v}) = 1$ . Thus  $(\bar{v}, H_i), (\bar{v}, H'_i) \in U(x_2)$ , which implies  $\Phi(x_2) \setminus St(\bar{v}) \subset \mathfrak{C}_1$ . Now  $\Phi(M_{i1}) \setminus St(\bar{v}) \subset \mathfrak{C}_1$  follows from the connectedness of  $M_{i1}$ . We can choose  $\mathfrak{C}_2$  in a similar way.

The above argument also implies for any vertex  $\bar{u}$  such that  $d(\bar{u}, \bar{v}) \neq 1$ ,  $\Phi(M_{i1}) \setminus St(\bar{u})$  is contained in a prime factor at  $\bar{u}$ . Note that if  $M_{i1} \subset St(\bar{v})$ , then  $\Phi(x) = \Phi(p(x))$  for any vertex  $x \in M_{i1}$ , hence  $M_{i2} = M_{i1}$ . Now we assume  $M_{i1} \not\subset St(\bar{v})$ , then we can set up as in Lemma 5.15 with respect to  $K = M_{i1}$ ,  $\bar{v} \in K$ ,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ . Let  $h_*$  and  $\bar{h}_*$  be the maps in Lemma 5.15. We claim  $\bar{h}_*(M_{i1}) = M_{i2}$ .

Pick vertex  $x \in M_{i1}$ . We claim for any vertex  $\bar{u}$  with  $d(\bar{u}, \bar{v}) \geq 1$ ,  $\Phi(x) \setminus St(\bar{u}) \neq \emptyset$  if and only if  $\bar{h}_*(\Phi(x)) \setminus St(\bar{u}) \neq \emptyset$ , and in this case  $\Phi(x) \setminus St(\bar{u})$  and  $\bar{h}_*(\Phi(x)) \setminus St(\bar{u})$  are in the same prime factor at  $\bar{u}$ . This follows from Lemma 5.15 (3b) when  $d(\bar{u}, \bar{v}) = 1$ . When  $d(\bar{u}, \bar{v}) > 1$ , note that  $\bar{v} \in \Phi(x)$  and  $\bar{v} \in \bar{h}_*(\Phi(x))$ , then the claim follows from Lemma 5.15 (3a). Thus for any  $(\bar{u}, H) \in U(x)$  with  $d(\bar{u}, \bar{v}) \geq 1$ ,  $\bar{h}_*(\Phi(x)) \subset H$ . Moreover, (1) of Lemma 5.15 implies  $\bar{h}_*(\Phi(x)) \setminus St(\bar{v}) \subset \mathfrak{C}_2$ , so  $\bar{h}_*(\Phi(x)) \subset \Phi(p(x))$  by our choice of  $\mathfrak{C}_2$  (recall that  $p : M_{i1} \rightarrow M_{i2}$  is the parallelism map). Denote the number of vertices in  $\Phi(x)$  by  $|\Phi(x)|$ , then  $|\Phi(x)| \leq |\Phi(p(x))|$ . By reversing the role of  $M_{i1}$  and  $M_{i2}$  and apply Lemma 5.15 with  $K = M_{i2}$ , we have  $|\Phi(p(x))| \leq |\Phi(x)|$ , hence  $|\Phi(x)| = |\Phi(p(x))|$ . But  $\bar{h}_*(\Phi(x))$  and  $\Phi(p(x))$  are both full subcomplexes, so  $\bar{h}_*(\Phi(x)) = \Phi(p(x))$ . Thus  $\bar{h}_*(\Phi(M_{i1})) = \Phi(M_{i2})$ .

Since  $\Phi(M_{ij})$  is a full subcomplex for  $j = 1, 2$ , we have  $\bar{h}_*(St(\bar{v}) \cap \Phi(M_{i1})) = \bar{h}_*(St(\bar{v}, \Phi(M_{i1}))) = St(\bar{v}, \Phi(M_{i2})) = St(\bar{v}) \cap \Phi(M_{i2})$ . However,  $\bar{h}_*$  fixes every point in  $St(\bar{v}) \cap \Phi(M_{i1})$ , so  $St(\bar{v}) \cap \Phi(M_{i1}) = St(\bar{v}) \cap \Phi(M_{i2})$ . Recall that  $\Phi(L_i) \cap \Phi(M_{i2}) \subset St(\bar{v})$ , so  $\Phi(M_{i1}) \cap St(\bar{v}) \subset \Phi(M_{i1}) \cap \Phi(M_{i2}) \subset \Phi(L_i) \cap \Phi(M_{i2}) \subset \Phi(M_{i2}) \cap St(\bar{v}) = \Phi(M_{i1}) \cap St(\bar{v})$  and all these sets are equal. Thus  $\Phi(L_{i+1})$  can be obtained by taking  $\Phi(L_i)$  and  $\Phi(M_{i1}) \cong \Phi(M_{i2})$ , and gluing them along  $St(\bar{v}, \Phi(M_{i1}))$ .

Let  $\Gamma'$  be the 1-skeleton of  $\Phi(L_1)$ . Note that  $L_1$  is one point and the above discussion implies the isomorphism type of  $\Gamma'$  does not depend on the choice of  $L_1$ .

**Step 3:** We show  $G(\Gamma')$  is a special subgroup of  $G(\Gamma)$  and  $G(\Gamma')$  is prime.

For convex subcomplex  $E \subset X(\Gamma')$ , let  $\{l_\lambda\}_{\lambda \in \Lambda}$  be the collection of standard geodesics in  $X(\Gamma', S)'$  such that  $l_\lambda \cap E \neq \emptyset$ . Denote the full subcomplex in  $\mathcal{P}(\Gamma')$  spanned by  $\{\Delta(l_\lambda)\}_{\lambda \in \Lambda}$  by  $\hat{E}$ . An edge  $e \subset X(\Gamma')$  is called a  $v$ -edge for  $v \in \mathcal{P}(\Gamma)$  if  $\Delta(l_e) = v$  ( $l_e$  is the standard geodesic containing  $e$ ). Similarly, an edge  $e \subset X$  is called a  $\bar{v}$ -edge for  $\bar{v} \in F(\Gamma)$  if the ultrafilters corresponding to two vertices of  $e$  differ by a  $\bar{v}$ -halfspace.

We are going to define a sequence of simplicial embeddings  $f_i : \Phi(L_i) \rightarrow \mathcal{P}(\Gamma')$  and cubical embeddings  $g_i : L_i \rightarrow X(\Gamma')$  with  $f_{i+1}|_{\Phi(L_i)} = f_i$  and  $g_{i+1}|_{L_i} = g_i$  which satisfy the following compatibility conditions:



- (1)  $g_i(L_i)$  is a compact convex subcomplex of  $X(\Gamma')$ .
- (2) For any vertex  $x \in L_i$ ,  $f_i(\Phi(x)) = \widehat{g_i(x)}$ . In particular,  $f_i(\Phi(L_i)) = \widehat{g_i(L_i)}$ .
- (3)  $g_i$  sends a  $\bar{v}$ -edge to a  $f_i(\bar{v})$ -edge.

We need several observations before the construction of  $g_i$  and  $f_i$ . Pick vertex  $v \in \mathcal{P}(\Gamma')$ , then the vertices of  $P_v$  are exactly those vertices  $x \in X(\Gamma')$  with  $v \in \hat{x}$ . Let  $l_v$  be a standard geodesic such that  $\Delta(l_v) = v$  and let  $h_v$  be a hyperplane dual to  $l_v$ . We identify  $P_v$  with  $l_v \times h_v$ . Then  $e \subset X(\Gamma')$  is a  $v$ -edge if and only if  $e \in P_v$  and  $e$  has trivial projection to the  $h_v$ -factor. Actually, these statements have their analogues in  $X$ .

Pick vertex  $x, x' \in X$  and let  $\{\bar{v}, H\} \in U(x)$  be a minimal halfspace. Then  $\bar{v} \subset \Phi(x)$ . We claim  $\bar{v} \in \Phi(x')$  if and only if for  $(\bar{v}', H', H'^c) \in \mathcal{W}(\Gamma)$  with  $(\bar{v}', H') \in U(x)$  and  $(\bar{v}', H'^c) \in U'(x')$ , we have  $d(\bar{v}, \bar{v}') \leq 1$ . The only if direction is clear. The other direction is true because if  $d(\bar{v}, \bar{v}') \leq 1$ , then  $\bar{v} \in St(\bar{v}') \subset H' \cap H'^c$ .

Let  $\mathcal{W}_{\bar{v}}(\Gamma)$  be the collection of  $\bar{v}'$ -walls with  $d(\bar{v}, \bar{v}') \leq 1$  and let  $\mathcal{H}_{\bar{v}}(\Gamma)$  be corresponding collection of halfspaces. Denote the corresponding  $CAT(0)$  cube complex by  $X_{\bar{v}}$ . Let  $\Sigma \subset \mathcal{H}(\Gamma)$  be the subset made of elements  $(\bar{w}, R)$  such that  $d(\bar{w}, \bar{v}) \geq 2$  and  $\bar{v} \in R$ . Pick an ultrafilter  $U_{\bar{v}}$  of  $\mathcal{H}_{\bar{v}}(\Gamma)$ , it is easy to see every pair of halfspaces in  $\Sigma \cup U_{\bar{v}}$  are compatible, thus  $\Sigma \cup U_{\bar{v}}$  is an ultrafilter of  $\mathcal{H}(\Gamma)$  and this induces a cubical embedding  $i_{\bar{v}} : X_{\bar{v}} \rightarrow X$ . Note that  $i_{\bar{v}}(X_{\bar{v}})$  is convex in  $X$  since two walls in  $\mathcal{W}_{\bar{v}}(\Gamma)$  are transverse in  $\mathcal{W}_{\bar{v}}(\Gamma)$  if and only if they are transverse in  $\mathcal{W}(\Gamma)$ . Since every  $\bar{v}$ -wall is transverse to all  $\bar{w}$ -walls with  $d(\bar{w}, \bar{v}) = 1$ ,  $X_{\bar{v}}$  admits a canonical splitting  $X_{\bar{v}} = h_{\bar{v}} \times [0, d_{\bar{v}} - 1]$ , here  $h_{\bar{v}}$  is isomorphic to the hyperplane in  $X$  corresponding to a  $\bar{v}$ -wall, and  $d_{\bar{v}}$  is the number of prime factors at  $\bar{v}$ . We will view  $X_{\bar{v}}$  as a convex subcomplex of  $X$ . Then vertices of  $X_{\bar{v}}$  are exactly those vertices  $x$  with  $\bar{v} \subset \Phi(x)$ . Moreover,  $e \subset X$  is a  $\bar{v}$ -edge if and only if  $e \in X_{\bar{v}}$  and  $e$  has trivial projection to the  $h_{\bar{v}}$ -factor.

Suppose we have already constructed  $g_i$  and  $f_i$ . Let  $e_i, h_i, N_i$  and  $(\bar{v}, H_i, H_i^c) = \Phi(h_i)$  be as in step 2 and let  $v = f_i(\bar{v})$ . Pick vertex  $x \in L_i$ . Then  $x \in X_{\bar{v}} \Leftrightarrow \bar{v} \in \Phi(x) \Leftrightarrow v \in f_i(\Phi(x)) \Leftrightarrow v \in \widehat{g_i(x)} \Leftrightarrow g_i(x) \in P_v$ . Thus  $g_i$  induced an isomorphism between  $X_{\bar{v}} \cap L_i$  and  $P_v \cap g_i(L_i)$ . Let  $X_{\bar{v}} \cap L_i = \bar{K}_i \times \bar{I}_i$  and  $P_v \cap g_i(L_i) = K_i \times I_i$  be the splitting induced from the splitting of  $X_{\bar{v}}$  and  $P_v$  as above ( $\bar{K}_i \subset h_{\bar{v}}$ ,  $K_i \subset h_v$ ,  $\bar{I}_i \subset [0, d_{\bar{v}} - 1]$  and  $I_i \subset l_v$ ). By (3),  $g_i|_{X_{\bar{v}} \cap L_i} = g_{i1} \times g_{i2}$  with  $g_{i1} : \bar{K}_i \rightarrow K_i$  and  $g_{i2} : \bar{I}_i \rightarrow I_i$ . Suppose  $\bar{I}_i = [0, a]$ , we identify  $I_i$  with  $[0, a]$  via  $g_{i2}$  and consistently identify  $l_v$  with  $\mathbb{R}$ .

Since  $e_i$  is a  $\bar{v}$ -edge,  $e_i \subset X_{\bar{v}}$ . We assume without loss of generality that  $x_i = e_i \cap L_i \in \bar{K}_i \times \{a\}$ . Then  $M_{i1} = L_i \cap N_i = \bar{K}_i \times \{a\}$  and  $N_i = \bar{K}_i \times [a, a+1]$ . Similarly,  $g_i(M_{i1}) = K_i \times \{a\}$ . Note that  $g_{i1}$  induces an isomorphism from  $\bar{K}_i \times [a, a+1]$  to  $K_i \times [a, a+1]$ , this defines  $g_{i+1} : L_{i+1} = L_i \cup N_i \rightarrow g_i(L_i) \cup (K_i \times [a, a+1])$ . Moreover,  $h_v \times [a, a+1]$ , which is the carrier of the hyperplane  $h_v \times \{a+1/2\}$ , satisfies  $(h_v \times [a, a+1]) \cap g_i(L_i) = K_i \times \{a\}$ , so  $g_i(L_i) \cup (K_i \times [a, a+1])$  is a compact convex subcomplex in  $X(\Gamma')$ .

We consider the left action  $G(\Gamma') \curvearrowright X(\Gamma')$  and let  $\alpha \in G(\Gamma')$  be the translation along  $l_v$  such that  $\alpha(K_i \times \{a\}) = K_i \times \{a+1\}$ . Then  $\alpha$  induces an isomorphism  $\alpha_* : \bar{K}_i \times \{a\} \rightarrow K_i \times \{a+1\}$ . It clear that  $\alpha_*(\hat{x}) = \widehat{\alpha(x)}$  for vertex  $x \in K_i \times \{a\}$  and  $\alpha$  sends  $v$ -edge to  $\alpha_*(v)$ -edge. We define  $f_{i+1}$  by

$$f_{i+1}(z) = \begin{cases} f_i(z) & \text{if } z \in \Phi(L_i) \\ (\alpha_* \circ f_i \circ (\bar{h}_*)^{-1})(z) & \text{if } z \in \Phi(M_{i2}) \end{cases}$$

Note that  $f_i(z) = (\alpha_* \circ f_i \circ (\bar{h}_*)^{-1})(z)$  for  $z \in \Phi(L_i) \cap \Phi(M_{i2}) = St(\bar{v}, \Phi(M_{i1}))$ , so  $f_{i+1}$  is well-defined. Now we show  $f_{i+1}$  and  $g_{i+1}$  satisfy the compatibility conditions (2) and (3). Since  $g_{i+1}|_{M_{i2}} = \alpha \circ g_i \circ p$ , ( $p : M_{i2} \rightarrow M_{i1}$  is the parallelism map), it suffices to check  $p$  and  $(\bar{h}_*)^{-1}$  satisfy the corresponding compatibility conditions. We have proved in step 2 that  $(\bar{h}_*)^{-1}(\Phi(x)) = \Phi(p(x))$  for vertex  $x \in M_{i2}$ . Let  $e \subset M_{i2}$  be a  $\bar{w}$ -edge. Then  $p(e)$  is also a  $\bar{w}$ -edge. We also deduce that  $d(\bar{w}, \bar{v}) = 1$ , hence  $(\bar{h}_*)^{-1}(\bar{w}) = \bar{w}$ . It follows that  $p$  sends  $\bar{w}$ -edge to  $(\bar{h}_*)^{-1}(\bar{w})$ -edge.

Let  $f : F(\Gamma) \rightarrow \mathcal{P}(\Gamma')$  and  $g : X \rightarrow X(\Gamma')$  be the simplicial embedding and the cubical embedding obtained by the above induction. Then  $E = g(X)$  is a compact convex subcomplex of  $X(\Gamma')$  and  $\hat{E} = f(F(\Gamma)) \cong F(\Gamma)$ . Thus  $G(\Gamma)$  is isomorphic to a special subgroup of  $G(\Gamma')$ . In particular,  $G(\Gamma')$  and  $G(\Gamma)$  are quasi-isometric, so  $\Gamma'$  is also of type II by Corollary 3.24. Next we show  $G(\Gamma')$  is actually prime.

**Lemma 5.17.** *Pick vertex  $x \in E$  and  $v \in \hat{x}$ . Let  $\bar{v} = f^{-1}(v)$ . Then*

$$\frac{\text{number of components in } \hat{x} \setminus St(v)}{\text{number of components in } \hat{E} \setminus St(v)} \leq \frac{1}{d_{\bar{v}}}.$$

*Recall that  $d_{\bar{v}}$  is the number of prime factors at  $\bar{v}$ .*

*Proof.* We use  $c(K)$  to denote the number of components in  $K$ . By consider the  $g$ -image of  $X_{\bar{v}} \cong h_{\bar{v}} \times [0, d_{\bar{v}} - 1]$ , we deduce that there is a segment  $I_v \subset E$  of length  $= d_{\bar{v}} - 1$  such that it is made of  $v$ -edges and it contains  $x$ . It follows from Corollary 3.14 that two vertices of  $\hat{I}_v$  are in the same  $v$ -branch if and only if they are in the same component of  $\hat{y} \setminus St(v)$  for some  $y \in I_v$ . Thus  $\hat{I}_v$  intersects at least  $d_{\bar{v}} \cdot c(\hat{x} \setminus St(v))$  many  $v$ -branches. The same is true for  $\hat{E}$  and the lemma follows.  $\square$

Let  $r : X(\Gamma') \rightarrow X(\Gamma)$  be the map in Theorem 2.15 and let  $r_* : \mathcal{P}(\Gamma') \rightarrow \mathcal{P}(\Gamma)$  be the induced simplicial isomorphism. Then  $r(E)$  is a vertex in  $X(\Gamma)$ . Moreover,  $r_*(\hat{E}) = r(\hat{E})$  by (1) and (2) of Theorem 2.15. Suppose there exists a non-prime vertex in  $\bar{u} \in F(\Gamma')$ . Let  $x \in E$  be a vertex and let  $v \in \hat{x}$  be the lift of  $\bar{u}$ . Then Lemma 5.17 implies the stretch factor of  $r_*$  at  $v$  (Lemma 5.4) is  $\geq d_{\bar{v}}$ , where  $\bar{v} = f^{-1}(v)$ . On the other hand, Lemma 5.4 implies that this stretch factor is bounded above by the number of prime factors at  $\pi \circ r_*(v)$ , which is equal to  $d_{\bar{v}}$  by Remark 5.10 (note that the composition  $F(\Gamma) \xrightarrow{f} \mathcal{P}(\Gamma') \xrightarrow{r_*} \mathcal{P}(\Gamma) \xrightarrow{\pi} F(\Gamma)$  is a simplicial isomorphism). Thus  $\bar{u}$  is prime, which is a contradiction.

In summary, we have proved the following result.

**Theorem 5.18.** *Let  $G(\Gamma)$  be a right-angled Artin group of type II. Then there exists a prime right-angled Artin group  $G(\Gamma')$  such that  $G(\Gamma)$  is isomorphic to a special subgroup of finite index in  $G(\Gamma')$ .*

**Remark 5.19.** Suppose  $\Gamma$  is of type II. For each vertex  $v \in \mathcal{P}(\Gamma)$ , we pick an identification  $f_v$  between the collection of  $v$ -sub-tiers and a copy of integers  $\mathbb{Z}_v$ . A  $v$ -halfspace is a subcomplex of  $\mathcal{P}(\Gamma)$  of form  $St(v) \cup f_v^{-1}([a, \infty))$  or  $St(v) \cup f_v^{-1}((-\infty, a])$ , where  $a \in \mathbb{Z}_v$ . We can put a pocset structure on the collection of all these halfspaces in a similar way as before. Then the  $CAT(0)$  cube complex associated with this pocset is isomorphic to  $X(\Gamma)$ .

We will not use this fact, so we will not give the detailed argument. However, it is instructive to think about the case when  $\text{Out}(G(\Gamma))$  is finite. Then the cube complex associated with the above pocset is actually isomorphic to  $X(\Gamma)$ . So the

quasi-isometry rigidity/flexibility of  $G(\Gamma)$  is reflected in how hard it is to reconstruct  $X(\Gamma)$  from  $\mathcal{P}(\Gamma)$  via cubulation.

It follows from Corollary 3.24, Theorem 5.18 and Theorem 5.9 that

**Theorem 5.20.** *If  $G(\Gamma_1)$  is a right-angled Artin group of type II, then  $G(\Gamma_2)$  is quasi-isometric to  $G(\Gamma_1)$  if and only if  $G(\Gamma_2)$  is commensurable to  $G(\Gamma_1)$ . Moreover, there exists a unique prime right-angled Artin group  $G(\Gamma)$  such that  $G(\Gamma_1)$  and  $G(\Gamma_2)$  are isomorphic to special subgroups of finite index in  $G(\Gamma)$ .*

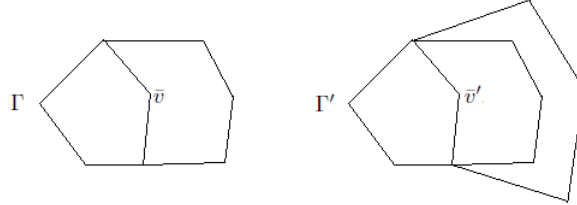
Now we discuss several applications of the above theorem.

**Example 5.21.** Let  $\Gamma_1$  be a 5-gon and let  $\Gamma_2$  be a 6-gon. We glue  $\Gamma_1$  and  $\Gamma_2$  along a vertex star to form  $\Gamma$  and claim  $\Gamma$  is prime. Let  $\bar{v}$  be the only vertex of  $\Gamma$  such that  $St(\bar{v})$  separates  $\Gamma$  and for  $i = 1, 2$  and let  $C_i = \Gamma_i \setminus St(\bar{v})$ . Pick  $v \in \mathcal{P}(\Gamma)$  such that  $\pi(v) = \bar{v}$  and let  $B_i$  be a  $v$ -branch such that  $\Pi(B_i) = C_i$ . It suffices to show  $B_1$  and  $B_2$  are not QII.

Suppose the contrary is true and let  $q$  be the quasi-isometry such that  $q_*(B_1) = B_2$ . By Corollary 3.14, there exist vertices  $x_1, x_2 \in P_v$  such that  $(\Gamma_i)_{x_i} \setminus St(v) \subset B_i$  for  $i = 1, 2$  ( $(\Gamma_i)_{x_i} \subset (\Gamma)_{x_i}$  is the lift of  $\Gamma_i$ ). By Remark 3.8 and Lemma 5.7,  $q_*((\Gamma_1)_{x_1}) = (\Gamma_1)_{x_3}$  for some vertex  $x_3 \in P_v$ . It follows from Lemma 3.9 that  $q_*(B_1) \neq B_2$ , which is a contradiction.

Theorem 5.20 implies that any  $G(\Gamma')$  quasi-isometric  $G(\Gamma)$  is isomorphic to a finite index subgroup of  $G(\Gamma)$ . Note that by the same proof, this statement is true in the case when  $\Gamma$  is obtained by gluing two distinct graphs  $\Gamma_1$  and  $\Gamma_2$  ( $\text{Out}(G(\Gamma_i))$  is finite for  $i = 1, 2$ ) along an isomorphic vertex star.

**Example 5.22.** We produce two graphs  $\Gamma$  and  $\Gamma'$  as below such that  $\mathcal{P}(\Gamma)$  and  $\mathcal{P}(\Gamma')$  are isomorphic, but  $G(\Gamma)$  and  $G(\Gamma')$  are not quasi-isometric.



First we show  $G(\Gamma)$  and  $G(\Gamma')$  are not quasi-isometric. Suppose the contrary is true and let  $q : G(\Gamma) \rightarrow G(\Gamma')$  be a quasi-isometry. Pick vertex  $v \in \mathcal{P}(\Gamma)$  such that  $\pi(v) = \bar{v}$  and let  $v' = q_*(v)$ . Then  $\pi(v') = \bar{v}'$ . This follows from the fact that  $\pi(v) = \bar{v}$  (or  $\pi(v') = \bar{v}'$ ) if and only if there are at least two QII classes among all the  $v$ -branches (or  $v'$ -branches). And this fact follows from the discussion in Example 5.21 and Corollary 3.14. However, the associated 2-tuple for  $\bar{v}$  and  $\bar{v}'$  are  $(1,1)$  and  $(1,2)$  respectively, which contradicts Lemma 5.4.

It remains to show  $\mathcal{P}(\Gamma)$  and  $\mathcal{P}(\Gamma')$  are isomorphic. Let  $f_1$  and  $f_2$  be two simplicial embeddings from  $\Gamma$  to  $\Gamma'$  such that (1) they cover different 6-gons in  $\Gamma'$ ; (2)  $f_1 = f_2$  when restricted to the 5-gon in  $\Gamma$ . We also use  $f_i$  to denote the group monomorphism induced by  $f_i$ . Let  $\omega \in G(\Gamma)$  be a geodesic word and write  $\omega = \omega_1 a_1 \cdots \omega_n a_n \omega_{n+1}$ , here  $a_i$  is a product of powers of elements in  $St(\bar{v})$  for all  $i$ , but  $\omega_i$  does not contain any powers of elements in  $St(\bar{v})$  ( $\omega_1$  or  $\omega_{n+1}$  may be trivial). By permuting letters in  $a_i$ , we have  $a_i = \bar{v}^{k_i} b_i$ , where  $b_i$  does not contain any power of  $\bar{v}$ .

Define a map  $h : G(\Gamma) \rightarrow G(\Gamma')$  by mapping  $\omega$  to  $\omega'_1 a'_1 \cdots \omega'_n a'_n \omega'_{n+1}$  such that (1)  $\omega'_i = f_1(\omega_i)$  if and only if  $k_{i-1}/2$  is an integer, otherwise  $\omega'_i = f_2(\omega_i)$ ; (2)  $a'_i = f_1(a_i)$  if and only if the first letter of  $w_{i+1}$  is inside the 5-gon, otherwise  $a'_i = \bar{v}^{\lfloor k_i/2 \rfloor} \cdot f_1(b_i)$ . Given a different geodesic word  $\omega_1 = \omega$ , we can obtain  $\omega_1$  from  $\omega$  by using the commutator relations to permute the letters in  $\omega$ , moreover, each word in the middle is also a geodesic word. Now it is easy to check that  $h$  is well-defined, and for each  $S$ -geodesic  $l \subset G(\Gamma)$ , there exists a unique  $S'$ -geodesic  $l' \subset G(\Gamma')$  such that  $h(l) = l'$  up to finite many points, moreover, if two  $S$ -geodesic are parallel (or orthogonal), then the corresponding  $h$ -images are parallel (or orthogonal), thus  $h$  induces a simplicial map  $h_* : \mathcal{P}(\Gamma) \rightarrow \mathcal{P}(\Gamma')$ . We can define a map  $h' : G(\Gamma') \rightarrow G(\Gamma)$  in a similar fashion which serves as the inverse of  $h$ , which would imply that  $h_*$  is actually a simplicial isomorphism.

## REFERENCES

- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [BJN10] Jason A. Behrstock, Tadeusz Januszkiewicz, and Walter D. Neumann. Quasi-isometric classification of some high dimensional right-angled Artin groups. *Groups Geom. Dyn.*, 4(4):681–692, 2010.
- [BKMM12] Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher. Geometry and rigidity of mapping class groups. *Geom. Topol.*, 16(2):781–888, 2012.
- [BKS08] Mladen Bestvina, Bruce Kleiner, and Michah Sageev. The asymptotic geometry of right-angled Artin groups. I. *Geom. Topol.*, 12(3):1653–1699, 2008.
- [BN08] Jason A Behrstock and Walter D Neumann. Quasi-isometric classification of graph manifold groups. *Duke Mathematical Journal*, 141(2):217–240, 2008.
- [CN05] Indira Chatterji and Graham Niblo. From wall spaces to CAT (0) cube complexes. *International Journal of Algebra and Computation*, 15(05n06):875–885, 2005.
- [CRSV10] Ruth Charney, Kim Ruane, Nathaniel Stambaugh, and Anna Vijayan. The automorphism group of a graph product with no SIL. *Illinois J. Math.*, 54(1):249–262, 2010.
- [CS11] Pierre-Emmanuel Caprace and Michah Sageev. Rank rigidity for CAT (0) cube complexes. *Geometric and functional analysis*, 21(4):851–891, 2011.
- [EF97] Alex Eskin and Benson Farb. Quasi-flats and rigidity in higher rank symmetric spaces. *Journal of the American Mathematical Society*, 10(3):653–692, 1997.
- [Ger98] V Gerasimov. Fixed-point-free actions on cubings. *Siberian Advances in Mathematics*, 8(3):36–58, 1998.
- [Hag08] Frédéric Haglund. Finite index subgroups of graph products. *Geometriae Dedicata*, 135(1):167–209, 2008.
- [Ham05] Ursula Hamenstaedt. Geometry of the mapping class groups III: Quasi-isometric rigidity. *arXiv preprint math/0512429*, 2005.
- [HP98] Frédéric Haglund and Frédéric Paulin. Simplicité de groupes d’automorphismes d’espaces à courbure négative. *Geometry and topology monographs*, 1, 1998.
- [Hua14a] Jingyin Huang. Quasi-isometry rigidity of right-angled artin groups i: the finite out case. *arXiv preprint arXiv:1410.8512*, 2014.
- [Hua14b] Jingyin Huang. Top dimensional quasiflats in  $cat(0)$  cube complexes. *arXiv preprint arXiv:1410.8195*, 2014.
- [HW08] Frédéric Haglund and Daniel T. Wise. Special cube complexes. *Geom. Funct. Anal.*, 17(5):1551–1620, 2008.
- [HW14] G Christopher Hruska and Daniel T Wise. Finiteness properties of cubulated groups. *Compositio Mathematica*, 150(03):453–506, 2014.
- [KK13] Sang-hyun Kim and Thomas Koberda. Embedability between right-angled Artin groups. *Geometry & Topology*, 17(1):493–530, 2013.
- [KL97] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 324(6):639–643, 1997.

- [Lau95] Michael R Laurence. A generating set for the automorphism group of a graph group. *Journal of the London Mathematical Society*, 52(2):318–334, 1995.
- [Nic04] Bogdan Nica. Cubulating spaces with walls. *Algebr. Geom. Topol*, 4:297–309, 2004.
- [Rol98] Martin Roller. Poc-sets, median algebras and group actions. *An extended study of Dunwoody’s construction and Sageev’s theorem*, 1998.
- [Sag95] Michah Sageev. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3)*, 71(3):585–617, 1995.
- [Sag12] Michah Sageev. CAT(0) cube complexes and groups. *IAS/Park City Mathematics Series*, 2012.
- [Ser89] Herman Servatius. Automorphisms of graph groups. *Journal of Algebra*, 126(1):34–60, 1989.